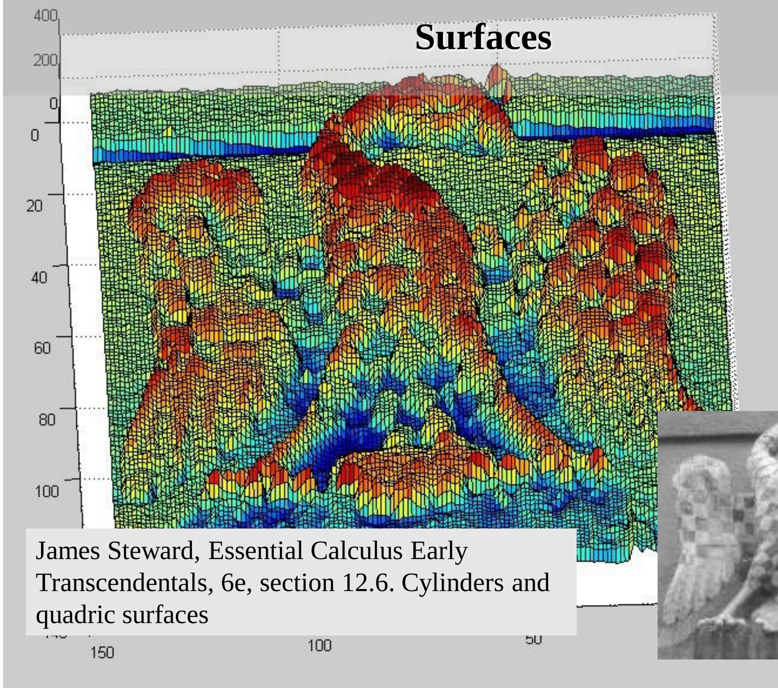
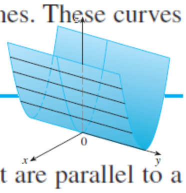
Surfaces



James Steward, Essential Calculus Early Transcendentals, 6e, section 12.6. Cylinders and quadric surfaces

In order to sketch the graph of a surface, it is useful to determine the curves of intersection of the surface with planes parallel to the coordinate planes. These curves are called **traces** (or cross-sections) of the surface.



CYLINDERS

A **cylinder** is a surface that consists of all lines (called **rulings**) that are parallel to a given line and pass through a given plane curve.

V EXAMPLE 1 Sketch the graph of the surface $z = x^2$.

SOLUTION Notice that the equation of the graph, $z = x^2$, doesn't involve y. This means that any vertical plane with equation $y = k$ (parallel to the xz-plane) intersects the graph in a curve with equation $z = x^2$. So these vertical traces are parabolas. Figure 1 shows how the graph is formed by taking the parabola $z = x^2$ in the xz-plane and moving it in the direction of the y-axis. The graph is a surface, called a

Parabolic cylinder

EXAMPLE 2 Identify and sketch the surfaces.

(a) $x^2 + y^2 = 1$ (b) $y^2 + z^2 = 1$

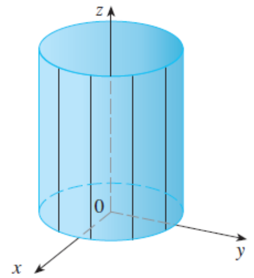


FIGURE 2 $x^2 + y^2 = 1$

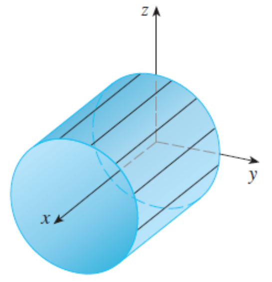


FIGURE 3 $y^2 + z^2 = 1$

■

⊘ **NOTE** When you are dealing with surfaces, it is important to recognize that an equation like $x^2 + y^2 = 1$ represents a cylinder and not a circle. The trace of the cylinder $x^2 + y^2 = 1$ in the xy-plane is the circle with equations $x^2 + y^2 = 1$, $z = 0$.

QUADRIC SURFACES

A **quadric surface** is the graph of a second-degree equation in three variables x, y, and z. The most general such equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

where $A, B, C, \ldots, J$ are constants, but by translation and rotation it can be brought into one of the two standard forms

$$Ax^2 + By^2 + Cz^2 + J = 0 \qquad \text{or} \qquad Ax^2 + By^2 + Iz = 0$$

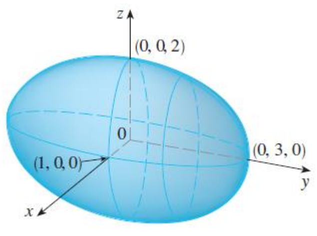


FIGURE 4

The ellipsoid $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$

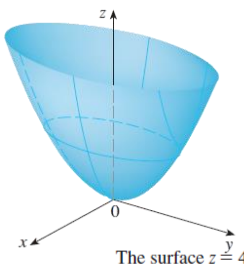


FIGURE 5

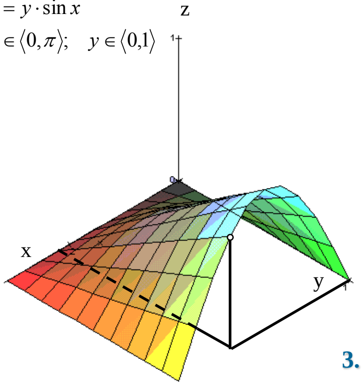
The surface $z = 4x^2 + y^2$ is an elliptic paraboloid. Horizontal traces are ellipses; vertical traces are parabolas.

Algebraic representation of the surfaces

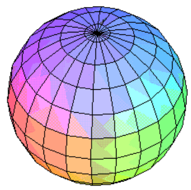
1. Explicit: z = f(x,y) ; [x,y] $\subset \Omega$

$z = y \cdot \sin x$

$x \in \langle 0, \pi \rangle; \quad y \in \langle 0, 1 \rangle$



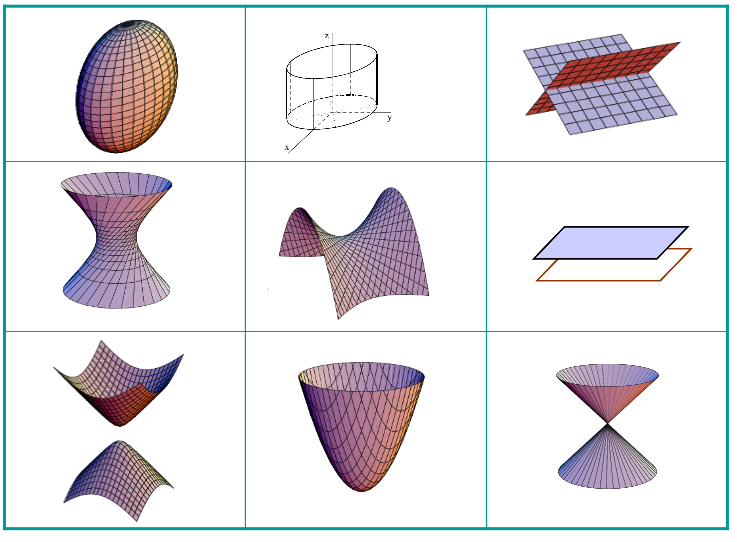
2. Implicit: F(x,y,z)=0



$x^2 + y^2 + z^2 = r^2$

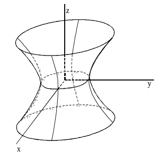
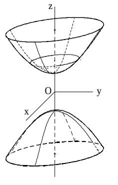
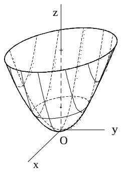
3. Parametric: P(*u*,*v*) = [x(*u*,*v*), *y*(*u*,*v*), *z*(*u*,*v*)]

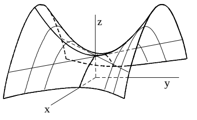
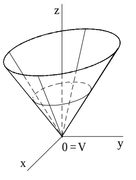
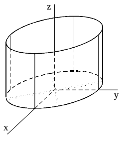
Quadric surfaces



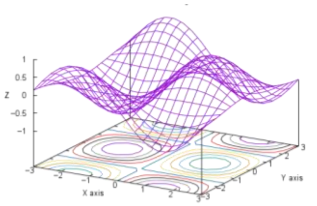
$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{13}yz + 2a_{14}x + 2a_{24}y + 2a_{34}z + a_{44} = 0$$

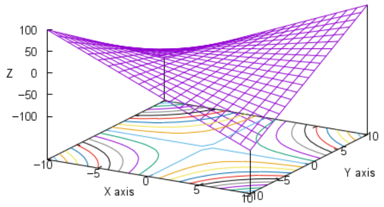
Quadric – algebraic surfaces of 2nd degree

Hyperboloid of one sheet $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$		$x = a\cosh u\cos t$ $y = b\cosh u\sin t$ $z = c\sinh u$
Hyperboloid of two sheets $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$		$x = a\cosh u$ $y = b\cos t\sinh u$ $z = c\sin t\sinh u$
Eliptic Paraboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}=z$		$x = at$ $y = bu$ $z = t^2+u^2$

Hyperbolic paraboloid $\frac{x^2}{a^2}-\frac{y^2}{b^2}=z$		$x = at$ $y = bu$ $z = t^2-u^2$
Cone $\frac{x^2}{a^2}+\frac{y^2}{b^2}=z^2$		$x = at\cos u$ $y = bt\sin u$ $z = t$
Cylinder $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$		$x = a\cos u$ $y = b\sin u$ $z = t$

Intersection curve of a plane and a quadric contour line



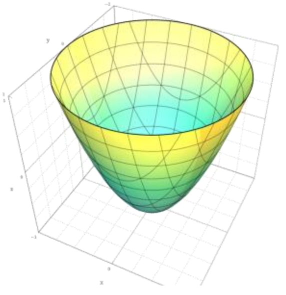


Hyperbolic paraboloid

$$z = x^2 - y^2$$

Cut by plane $z = z_0$; $z_0 = x^2 - y^2$

$$C(t) = \left(\sqrt{z_0}\,.\cosh t, \sqrt{z_0}\,.\sinh t, z0\right);\ z \neq z_0$$



Circular paraboloid

$$z = x^2 + y^2$$

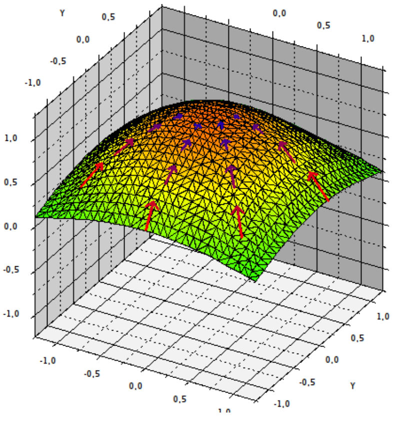
Cut by plane $z = z_0$; $z_0 = x^2 + y^2$

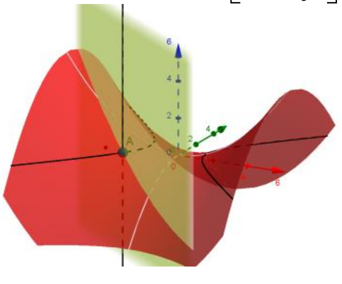
$$C(t) = \left(\sqrt{z_0}\,.\cos t, \sqrt{z_0}\,.\sin t, z0\right)$$

Gradient ∇f of the scalar field f(x,y)

- The gradient points in the direction of the greatest rate of increase of the function, and its magnitude is the slope of the graph in that direction.
- A level curve (or contour line), is the set of all points where some function has a given value.

$$\nabla f(x, y) = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right]$$





05_gradient.ggb

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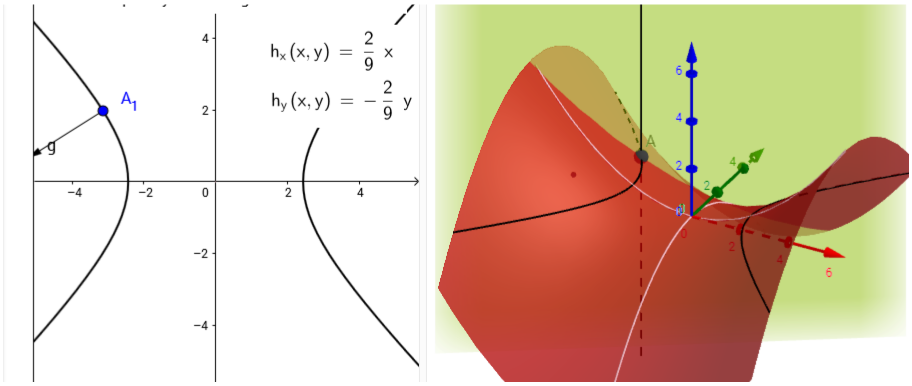
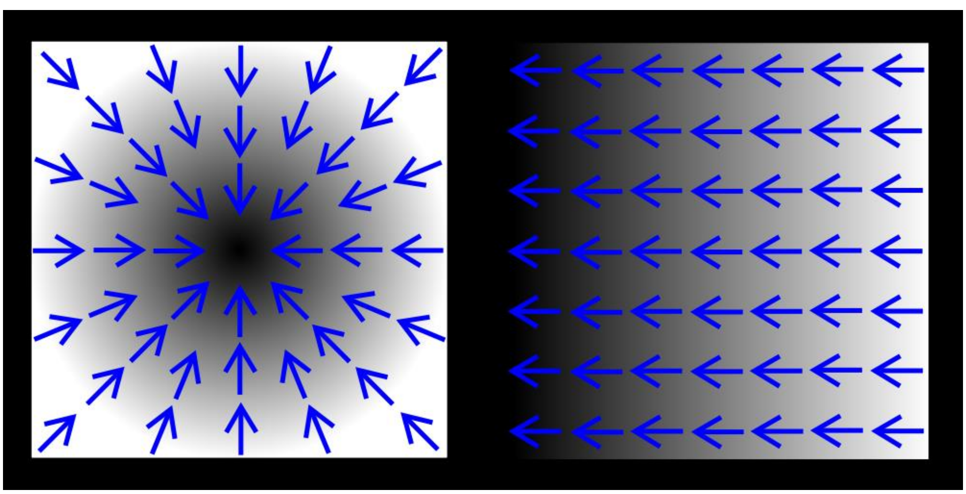
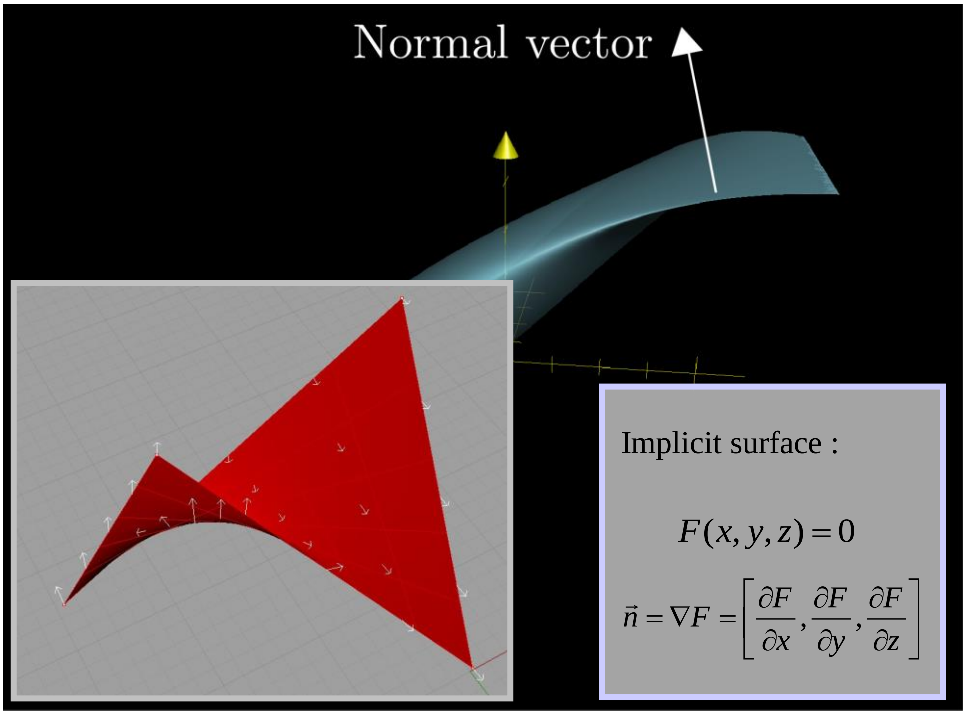


Gradient ∇f
of the function f(x,y)



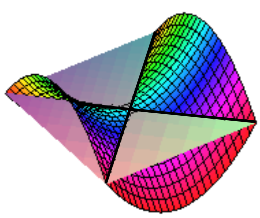


Implicit surface :

$$F(x, y, z) = 0$$

$$\vec{n} = \nabla F = \left[\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right]$$

Hyperbolic paraboloid

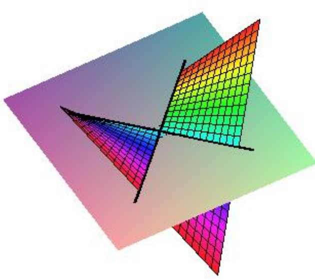


$z=x^2 - y^2$

Tangent plane at the point

X(0,0) = [0,0,0]

$$\tau : z = 0$$



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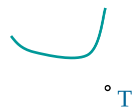
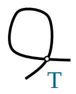
Hyperbolic, elliptic and parabolic points on the surface

Point T is calling:

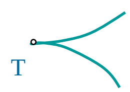
- Elliptic point
  T is isolated (or the only) point of the cutting curve

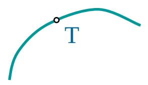


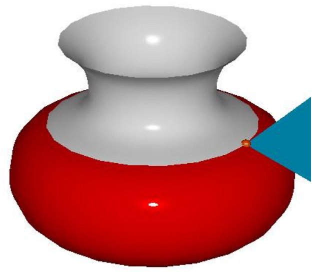
- Hyperbolic point
  T is nodal point



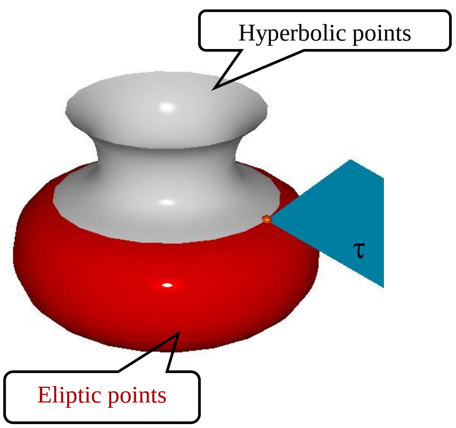
- Parabolic point



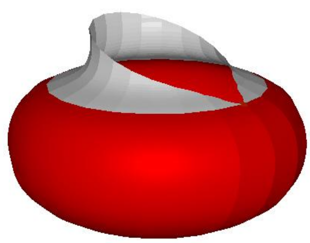




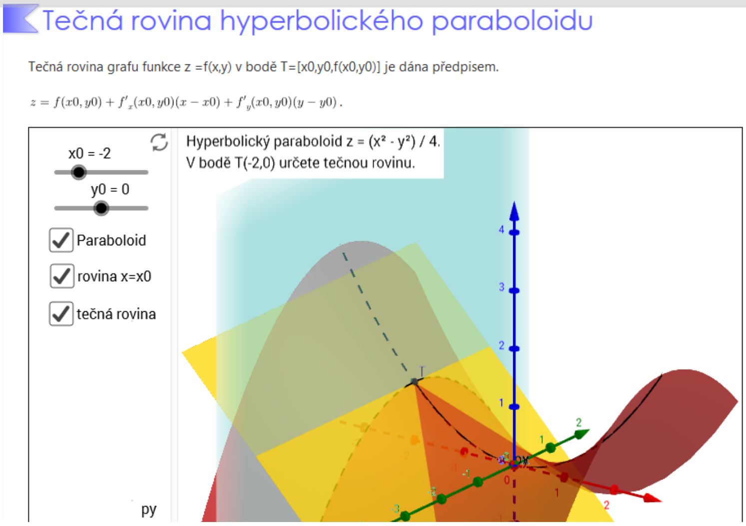
Loci of hyperbolic and elliptic points are separated by curve of parabolic points



Surface cut off by tangent plane
in parabolic point



Tangent plane of the hyperbolic paraboloid



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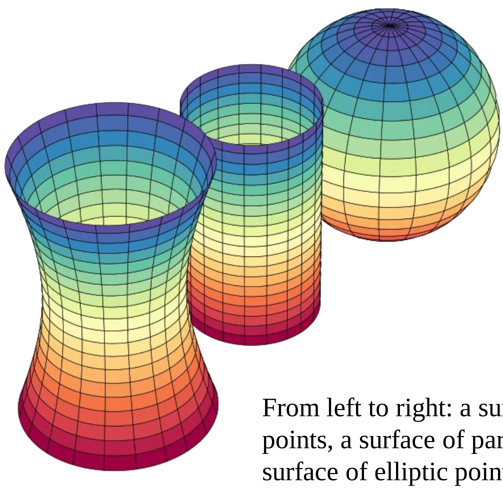
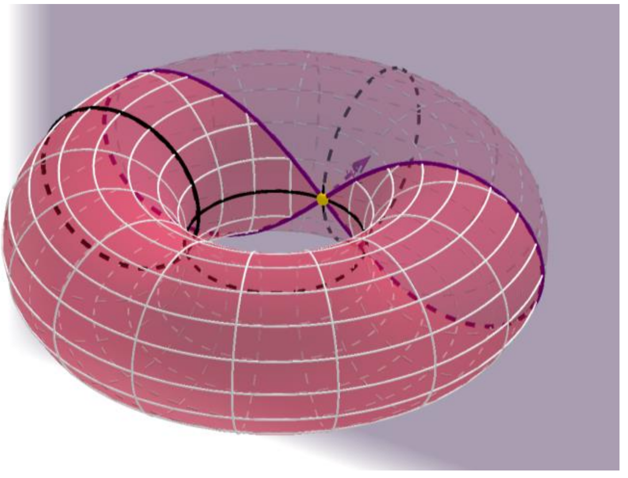
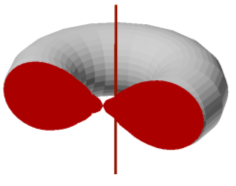


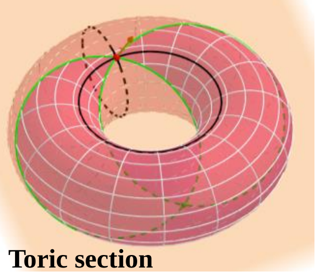
From left to right: a surface of hyperbolic points, a surface of parabolic points, and a surface of elliptic points.

Lemniscate as a toric section



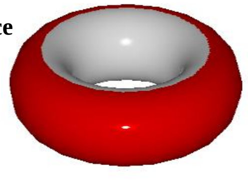
Torus is a surface of revolution generated by revolving a circle in three-dimensional space about an axis coplanar with the circle. If the axis of revolution does not touch the circle, the surface has a ring shape and is called a torus of revolution.

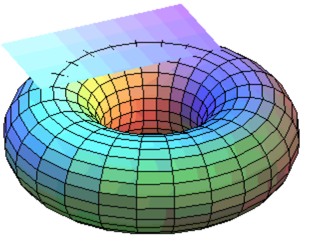




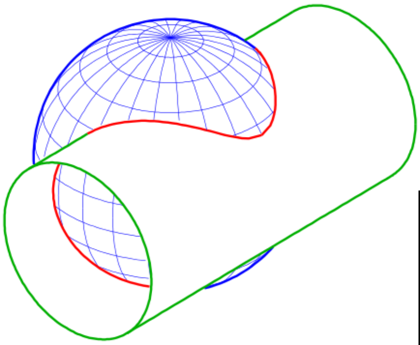
Toric section

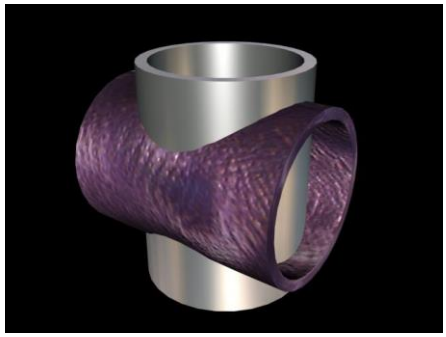
Elliptic (red) and hyperbolic (grey) point on the surface





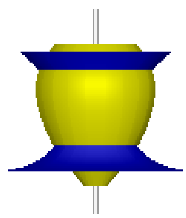
Intersection curves



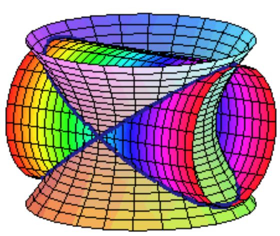


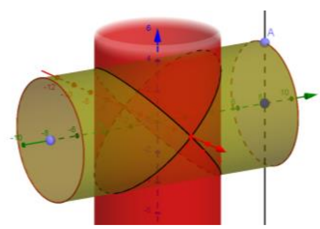
Intersection of two surfaces

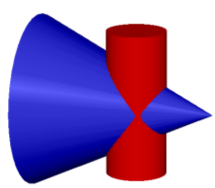
Hyperboloid of one sheet and cylinder



$$x^2 + y^2 - z^2 = 1$$
$$x^2 + z^2 = 1$$

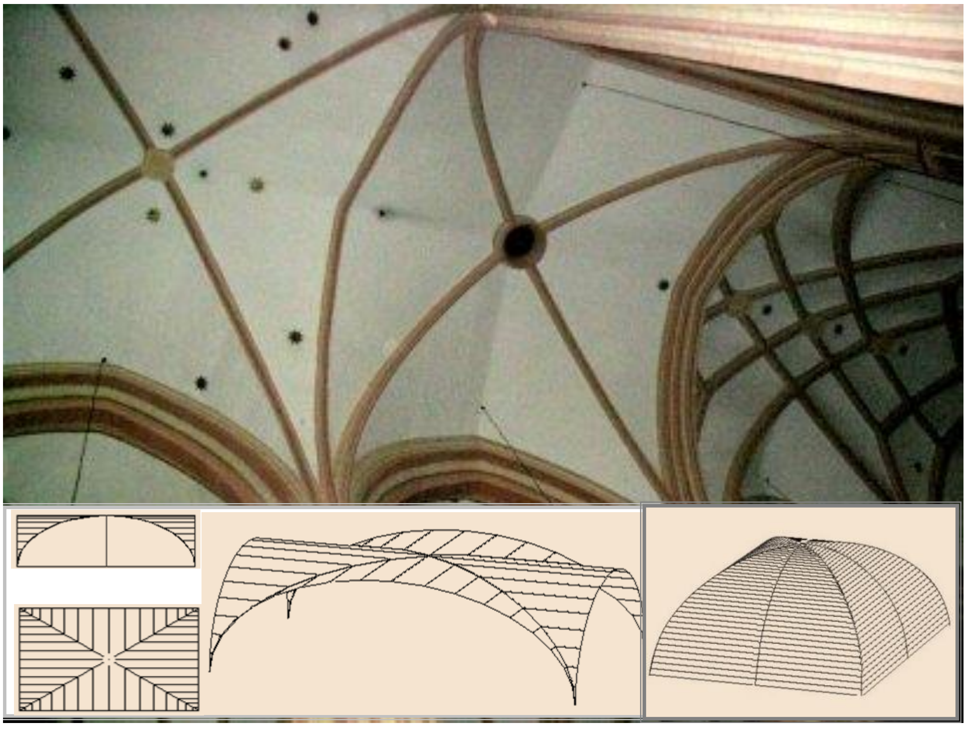






Parametric equation of the intersection curves

$$X(t) = \left[\cos(t), \pm\sqrt{2}\sin t, \sin t\right]$$



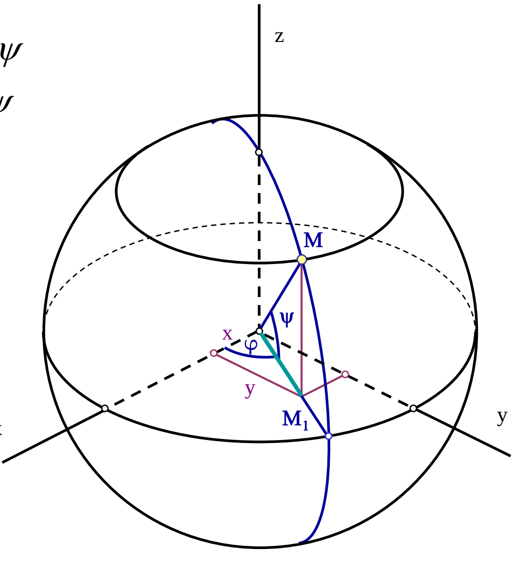
Parameterizing sphere with longitude and latitude

$$|OM_1| = d$$

$$x = d \cdot \cos\varphi \qquad d = r \cdot \cos\psi$$

$$y = d \cdot \sin\varphi \qquad z = r \cdot \sin\psi$$

$$x = r \cdot \cos\psi \cdot \cos\varphi$$
$$y = r \cdot \cos\psi \cdot \sin\varphi$$
$$z = r \cdot \sin\psi;$$
$$\varphi \in \langle 0, 2\pi); \quad \psi \in \left\langle -\frac{\pi}{2}, \frac{\pi}{2} \right\rangle$$

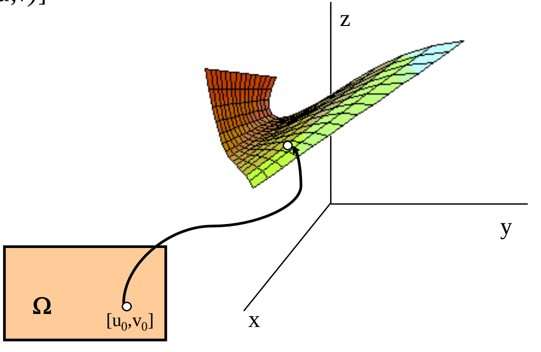


Parametric surface

Surface is two parameters set of points $P(u,v)$, whose coordinates could be expressed by continuous transformation $\Omega \to E^3$, $[u, v] \to [x, y, z]$

$P(u,v) = [x(u,v); y(u,v); z(u,v)]$

$x = x(u,v)$
$y = y(u,v)$
$z = z(u,v); [u, v] \in \Omega$



Circular Cylinder

Implicit equation:
$x^2+y^2 = r^2$

Parametric form:
$x = r \cos u$
$y = r \sin u$
$z = v$, $u \in <0, 2\pi>$, $v \in <0, h>$

Curve on the surface:
$u = t, v = t, \quad t \in <0, 4\pi>$

$x = r \cos t$
$y = r \sin t$
$z = t$, $t \in <0, 2\pi>$