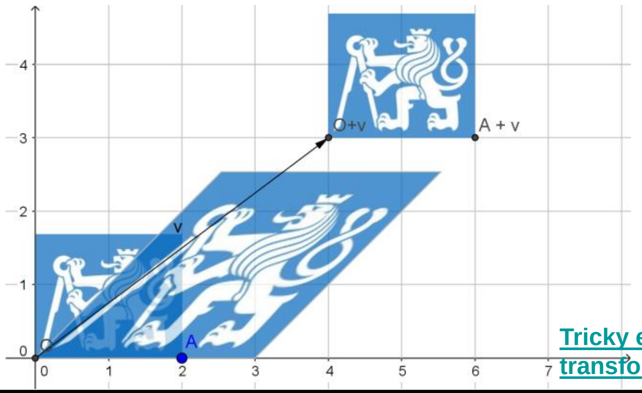
Linear and affine transformations

- Linear Algebra Review
  - Matrices
  - Transformations
- Affine transformations in Euclidean space

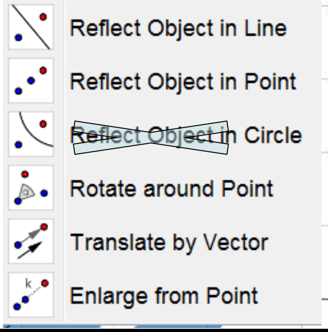


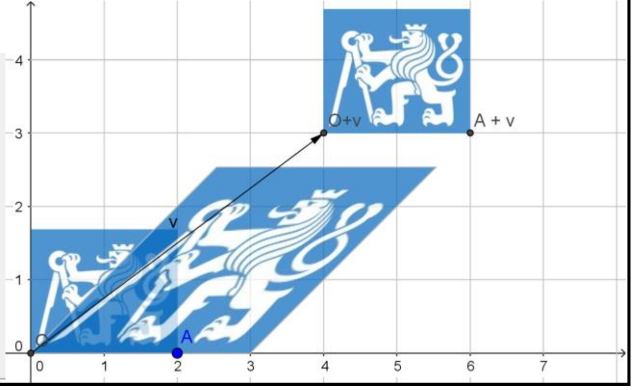
[Tricky examples of nonlinear transformations (Youtube)]

1

Geometric transformations

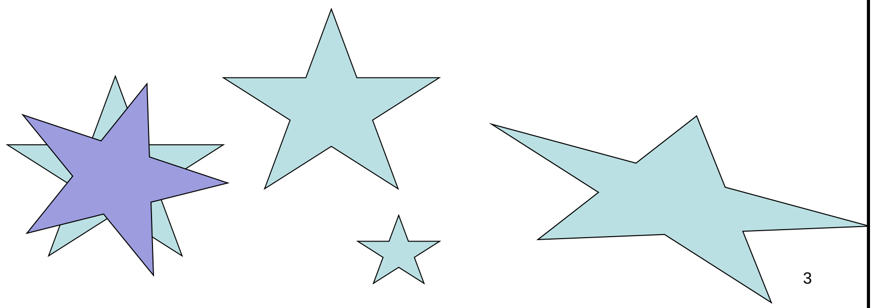
- Geometric transformations map points in one space to points in another: *(x',y',z') =* ***f****(x,y,z), i.e. in vector form X' = f(X)*
- These transformations can be very simple, such as scaling each coordinate, or complex, such as non-linear twists and bends.
- We'll focus on transformations that can be represented easily with matrix operations.
- We'll start in 2D...





2D Affine Transformations

- An *affine transformation* is any transformation that preserves co-linearity (i.e., all points lying on a line initially still lie on a line after transformation) and ratios of distances (e.g., the midpoint of a line segment remains the midpoint after transformation).



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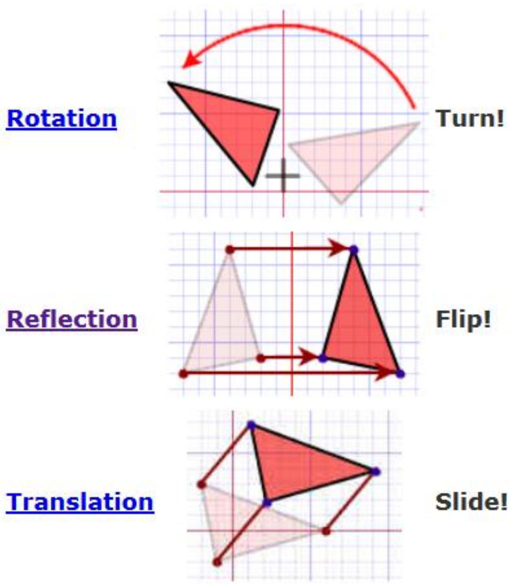
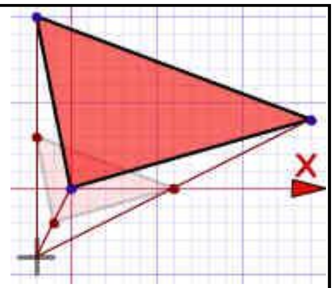
Isometry (Congruence)
Preserves Lengths



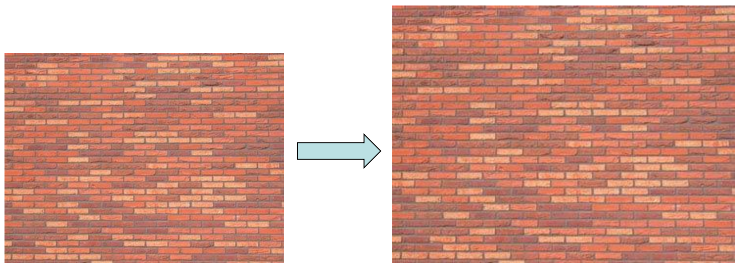
After any of those transformations (turn, flip or slide),
the shape still has the same size, area, angles and line lengths.

Similarity (Resizing)
Preserves angles

When you resize a shape it gets bigger or smaller.
... but it still looks *similar:*
all angles are the same



The face and body are still in proportion



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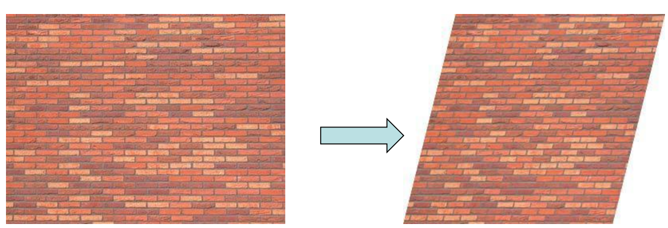
Affinity
Preserves parallel line and dividing proportion

Affine transformation preserves parallelism, dividing proportion, linearity and incidence.

Similarity and congruence can be viewed as a special case of the affine transformation.



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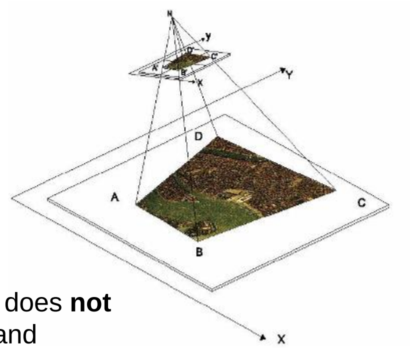
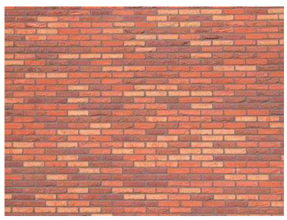
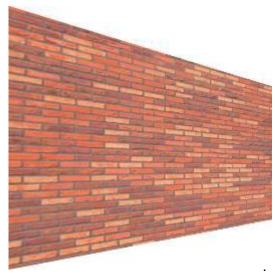
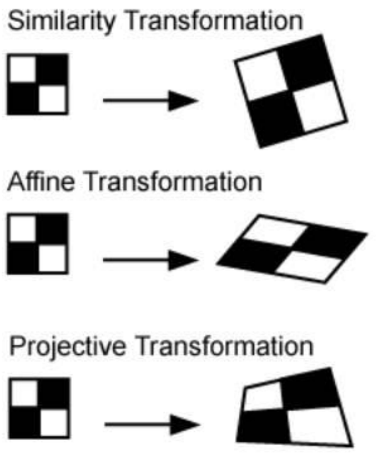
HOMOGRAPHY (PROJECTIVE)



The projective transformation does **not** preserve parallelism, length, and angle. But it still preserves collinearity and incidence.





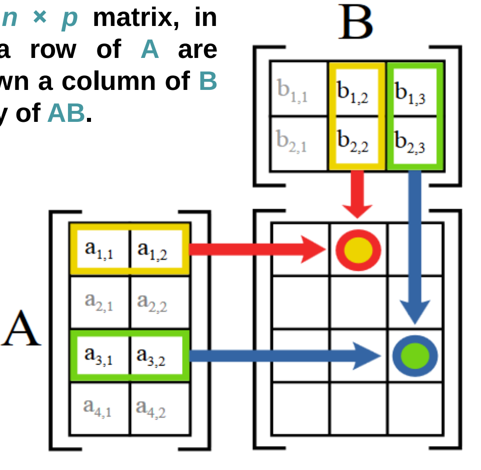


Linearity	Parallel	Angle	Length
x	x	x	
x	x		
x			

Matrix Multiplication

if A is an $n \times m$ matrix and B is an $m \times p$ matrix, their matrix product AB is an $n \times p$ matrix, in which the m entries across a row of A are multiplied with the m entries down a column of B and summed to produce an entry of AB.

A . B



Matrix Multiplication is not commutative

AB ≠ BA

When we change the order of multiplication, the answer is (usually) **different**.

Example:

See how changing the order affects this multiplication:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 10 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 7 & 10 \end{bmatrix}$$

The answers are different!

Expand vector notation for linear equations

$$\vec{x}' = A\vec{x}$$

$$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \vec{x}' = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}$$

Solution

$$\vec{x}' = A\vec{x}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$x' = 2x + 1y$$

$$y' = 0x - y$$

- **The linear transformation given by a matrix**

Let *A* be an 2 × 2 matrix. The function *T* defined by

$$T(\mathbf{v}) = A\mathbf{v}$$

is a linear transformation from R^2 into R^2.

- **Note:**

$$\vec{x}' = A\vec{x}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$x' = ax + by$$

$$y' = cx + dy$$

- **The linear transformation given by a matrix**

Let *A* be an *m*×*n* matrix. The function *T* defined by

$$T(\mathbf{v}) = A\mathbf{v}$$

is a linear transformation from R^n into R^m.

- **Note:**

R^n vector $\quad$ R^m vector

$$A\mathbf{v} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{bmatrix}$$

$$T(\mathbf{v}) = A\mathbf{v}$$

$$T : R^n \longrightarrow R^m$$

- **Two representations of the linear transformation $T: R^3 \to R^3$:**

$$(1) T(x_1, x_2, x_3) = (2x_1 + x_2 - x_3, -x_1 + 3x_2 - 2x_3, 3x_2 + 4x_3)$$

$$(2) T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 3 & -2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- **Three reasons for matrix representation of a linear transformation:**
  - **It is simpler to write.**
  - **It is simpler to read.**
  - **It is more easily adapted for computer use.**

Representation of 2D linear map

- We can represent a **2-D transformation M** by a matrix

$$\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- If **x** is a column vector, *M* goes on the left: $\mathbf{x}' = \mathbf{M}\mathbf{x}$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- If **x** is a row vector, M^T goes on the right: $\mathbf{x}' = \mathbf{x}\mathbf{M}^{\mathrm{T}}$

$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

- We will use **column vectors**.

Property of Linear Transforms

- Basis vectors map to columns of matrix.
- Origin (0,0) is always fixed point.
- Composition of M and M^{-1} gives identity.
- Determinant det(M) is scaling factor of the linear transformation described by the matrix M.

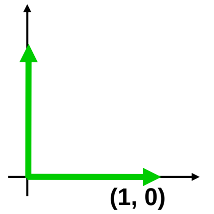
$$\mathbf{x}' = \mathbf{M}\bullet\mathbf{x}$$

$$\mathbf{M}\bullet\mathbf{x} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$

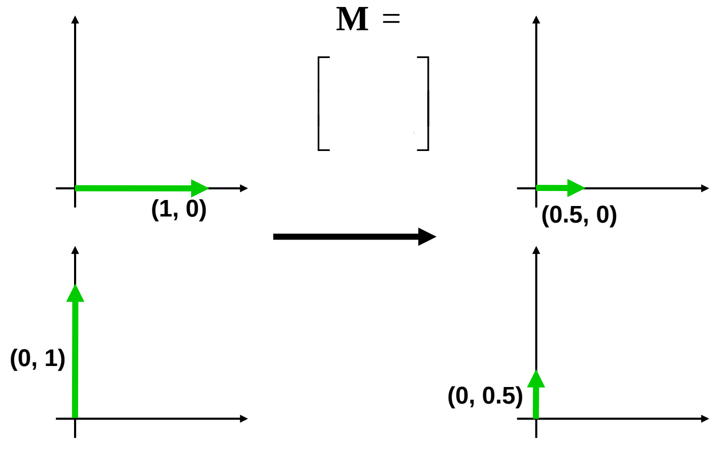
$$\mathbf{x}' = \mathbf{M}\mathbf{x}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$

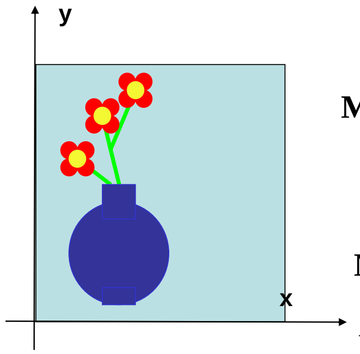


Scaling by 0.5 $\quad \mathbf{x}' = \mathbf{M}\bullet\mathbf{x}$



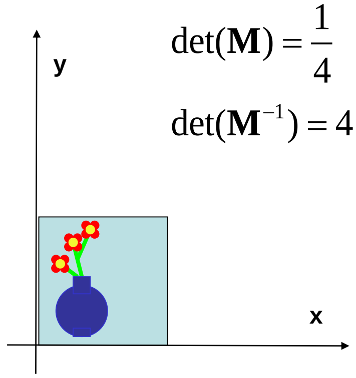
Scaling by 0.5

$$\mathbf{x}' = \mathbf{M}\bullet\mathbf{x}$$



$$\mathbf{M} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$$\mathbf{M}^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$



$$\det(\mathbf{M}) = \frac{1}{4}$$

$$\det(\mathbf{M}^{-1}) = 4$$

Inverse mapping = scaling by 2

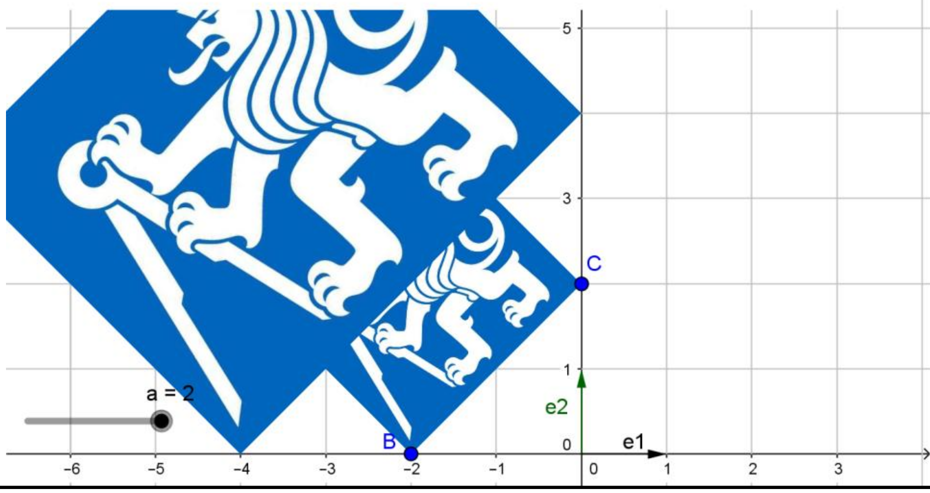
Composition of M and M^{-1} gives identity. Determinant is scaling factor of the linear transformation described by the matrix.

Homothety - Scaling

Describe the transformation represented by matrix S = {{2, 0}, {0, 2}}.

Find all fixed points and directions.

List all invariants.



Homothety - Scaling

Describe the transformation represented by matrix
S = {{2,0},{0,2}}.
Find all fixed points and directions. List all invariants.

$$S = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Fixed points: $X' = X;\ X' = SX$

$X = S \cdot X$

$$\left.\begin{array}{l} x = 2x \\ y = 2y \end{array}\right\} x = 0, y = 0$$

Only one fixed point (0, 0).

Fixed directions: $\mathrm{v}' = \lambda v;\ v' = Sv$

$\lambda v = S \cdot v$

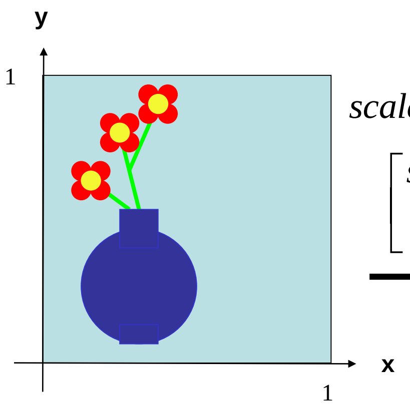
$$\left.\begin{array}{l} \lambda x = 2x \\ \lambda y = 2y \end{array}\right\} \Rightarrow \begin{array}{l} x(\lambda - 2) = 0 \\ y(\lambda - 2) = 0 \end{array} \Rightarrow$$

$$\Rightarrow \lambda = 2, x \in R, y \in R$$

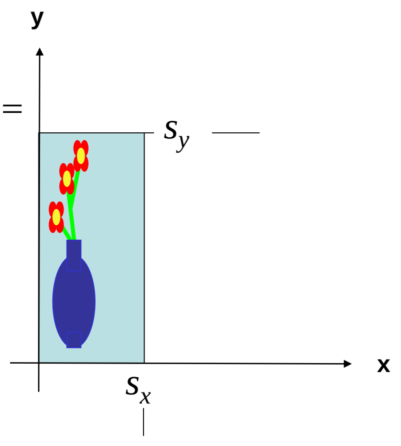
All directions are fixed.

General Scaling

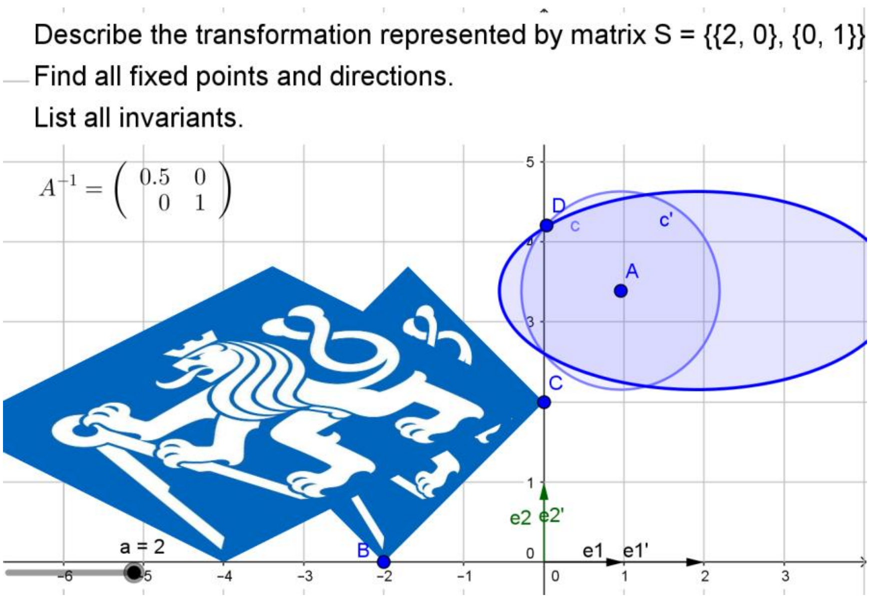
$$\mathbf{x}' = scale(s_x, s_y) \bullet \mathbf{x}$$



$$scale(s_x, s_y) = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$



General Scaling



General Scaling

Describe the transformation represented by matrix
S = {{2,0},{0,**1**}}.
Find all fixed points and directions. List all invariants.

$$S = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

Fixed points: $X' = X; X' = SX$

$X = S \cdot X$

$$\left.\begin{aligned} x &= 2x \\ y &= y \end{aligned}\right\} x = 0, y \in R$$

$\mathrm{FP} = \{(0,t), t \in R\}$.

Fixed directions: $\mathrm{v}' = \lambda v; v' = Sv$

$\lambda v = S \cdot v$

$$\left.\begin{aligned} \lambda x &= 2x \\ \lambda y &= y \end{aligned}\right\} \Rightarrow \begin{aligned} x(\lambda - 2) &= 0 \\ y(\lambda - 1) &= 0 \end{aligned} \Rightarrow$$

$\Rightarrow \lambda = 2, x \in R, y = 0;\ fd = (t,0)$

$\Rightarrow \lambda = 1, x = 0, y \in R;\ fd = (0,t)$

2 fixed directions: [(1,0)] and [(0,1)].

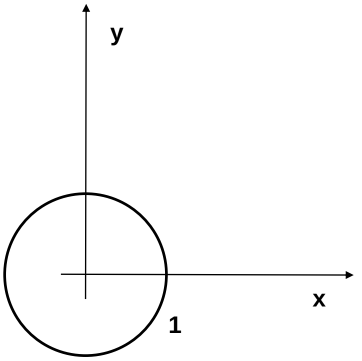
Scaling of a circle

$$\mathbf{x}' = M \bullet \mathbf{x}$$

$$\mathbf{x} = M^{-1}\mathbf{x}'$$

$$x = \frac{x'}{a} \qquad x^2 + y^2 = 1$$

$$y = \frac{y'}{b} \qquad \left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 = 1$$

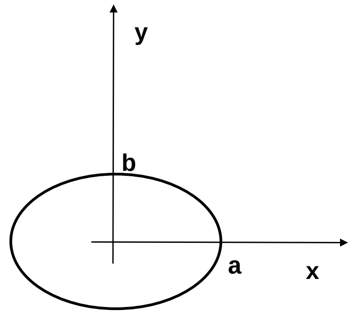


$$M = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

$$\longrightarrow$$

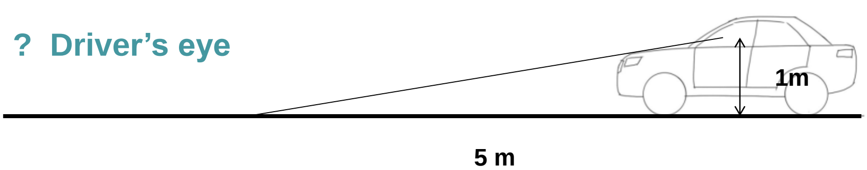
$$\longleftarrow$$

$$M^{-1} = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix}$$



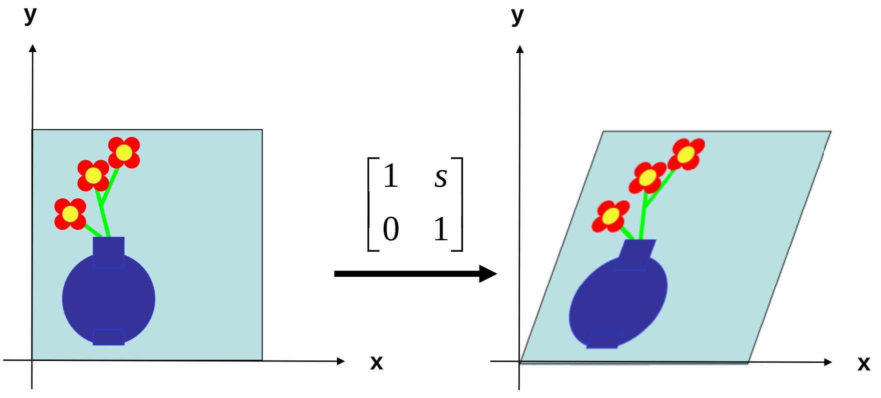
Real image

? Driver's eye



Shear-x

$$\mathbf{x}' = \mathbf{M} \bullet \mathbf{x}$$

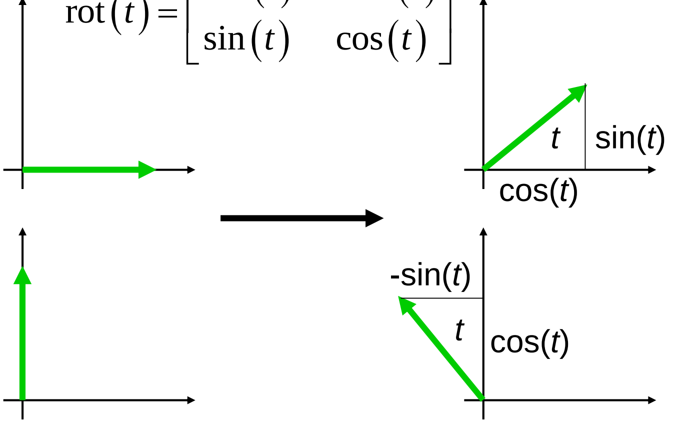


$$\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$

Rotation

$$\mathbf{x}' = R(t) \bullet \mathbf{x}$$

$$\operatorname{rot}(t) = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}$$



Trajectory of point A = (r, 0) in revolution

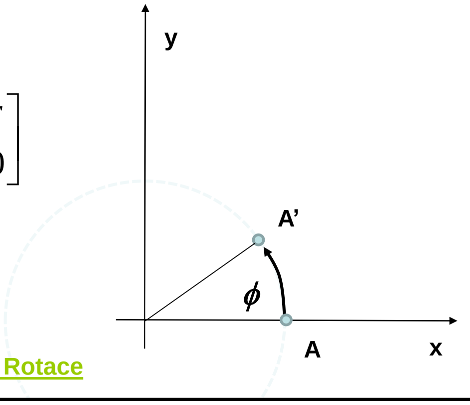
$$\mathbf{x}' = R\bullet\mathbf{x}$$

$$R = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix};\ \det(R) = 1$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix} \begin{bmatrix} r \\ 0 \end{bmatrix}$$

$$x' = r\cdot\cos\phi$$

$$y' = r\cdot\sin\phi$$



GeoGebra book 2.1 Rotace

Exercise: Rotation

$$\mathbf{x}' = R(\phi)\bullet\mathbf{x}$$

Estimate parameter *a* so that matrix B represents revolution about origin.
Find all fixed points and directions.

$$B = \begin{bmatrix} a & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & a \end{bmatrix} \qquad R(\phi) = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$$

1. method: comparing elements *R* and *B*.

$$\sin\phi = \frac{\sqrt{2}}{2} \Rightarrow a = \cos\phi = \frac{\sqrt{2}}{2}$$

2. method:

$$\det(B) = 1 \Rightarrow a^2 + \frac{2}{4} = 1$$

Matrix Representation of rotation

Rotation

$$B=\begin{bmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix}$$

Find all fixed points.

$$\mathbf{x}'=B\cdot\mathbf{x},\ \mathbf{x}'=\mathbf{x}$$
$$\mathbf{x}=B\cdot\mathbf{x}\Rightarrow(B-E)\mathbf{x}=o$$

GeoGebra tool `ReducedRowEchelonForm(M)` eliminates non diagonal elements by row operations (= Gaussian elimination).

$$(B-E)\sim\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

Using back-substitution, unknowns x, y can be solved for. Solution x = 0 and y = 0 gives only one fixed point FP = (0,0).

Rotation

$$B=\begin{bmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix}$$

Find all fixed directions.

$$\mathbf{x}'=B\cdot\mathbf{x},\ \mathbf{x}'=\lambda\mathbf{x}$$
$$\lambda\mathbf{x}=B\cdot\mathbf{x}\Rightarrow(B-\lambda E)\mathbf{x}=o$$

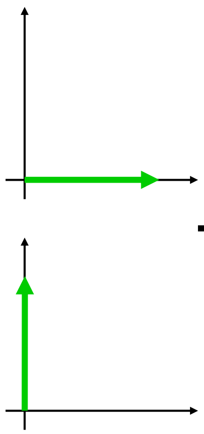
Matrix `(B-λE)` must be singular for non trivial solutions **x**, but `Det(B-λE)=0` has no real solution.

$$|B-\lambda E|=\begin{vmatrix}\frac{\sqrt{2}}{2}-\lambda & -\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}-\lambda\end{vmatrix}=0$$

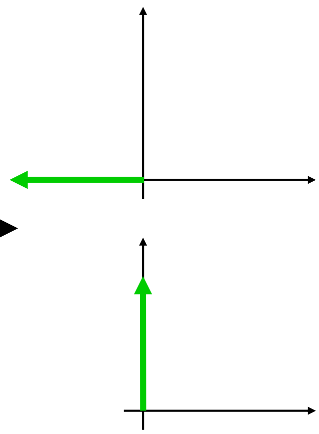
General rotation hasn’t fixed directions.

Reflection in y-axis

$$\mathbf{x}' = ref_y \bullet \mathbf{x}$$

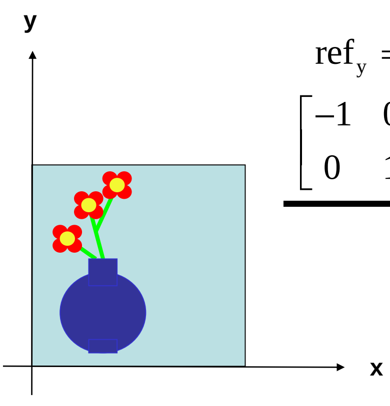


$$\mathrm{ref}_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

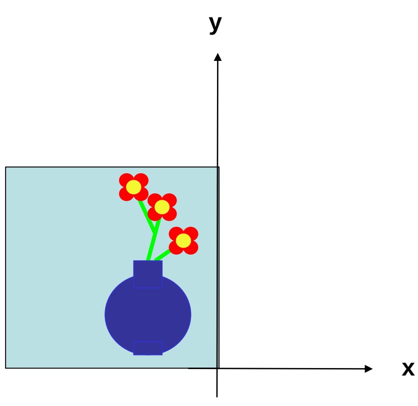


Reflection in y-axis

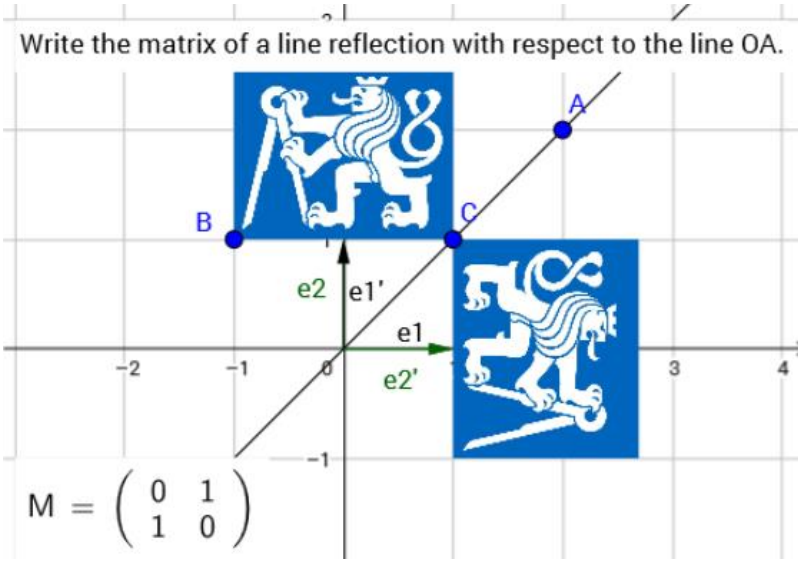
$$\mathbf{x}' = ref_y \bullet \mathbf{x}$$



$$\mathrm{ref}_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$



Line reflection

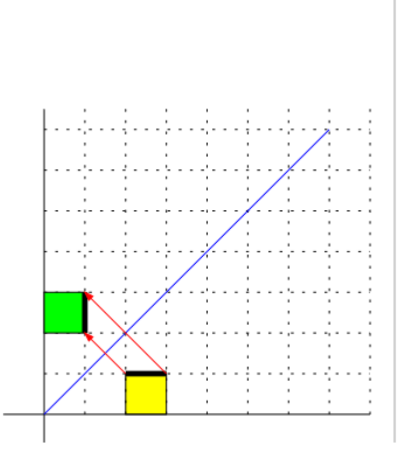


Reflection in the line y= x

Reflection in the line $y = x$ boils down to swapping x- and y-coordinates:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$$

Q: what is the inverse of this matrix?



Composing Linear Transformations

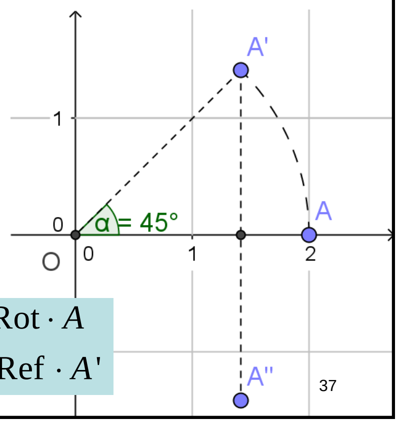
$$T_1(\mathbf{v}) = M_1 \cdot \mathbf{v}$$
$$T_2(\mathbf{v}) = M_2 \cdot \mathbf{v}$$

- If T_1 and T_2 are transformations
  - $T_2\, T_1(\mathbf{v}) =_{\text{def}} T_2(\, T_1(\mathbf{v}))$
- If T_1 and T_2 are represented by matrices M_1 and M_2
  - $T_2\, T_1$ is represented by $M_2\, M_1$
  - $T_2\, T_1(\mathbf{v}) = T_2(\, T_1(\mathbf{v})) = (M_2\, M_1)(\mathbf{v})$

- **Order is important!**

  reflect(x) (rot(0,$\boldsymbol{\alpha}$)): A → A' → A''

  rot(0,$\boldsymbol{\alpha}$) (reflect(x)): A → A → A'



$$A' = \text{Rot} \cdot A$$
$$A'' = \text{Ref} \cdot A'$$

Composing Linear Transformations

- **Order is important!**

  reflect(x) (rot(0,$\boldsymbol{\alpha}$)): A → A' → A''

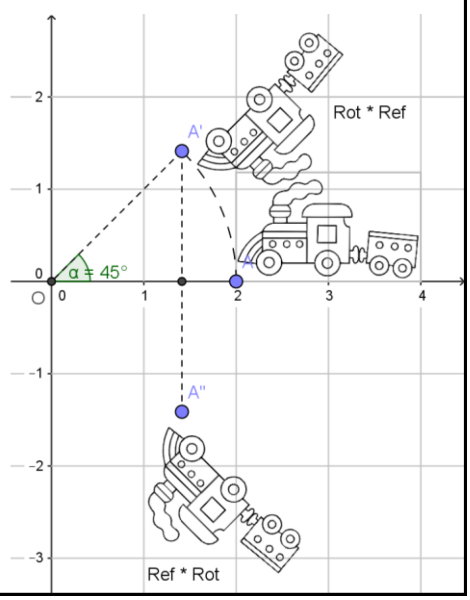
  rot(0,$\boldsymbol{\alpha}$) (reflect(x)): A → A → A'

$$A' = \text{Rot} \cdot A \qquad A'' = \text{Ref} \cdot A'$$

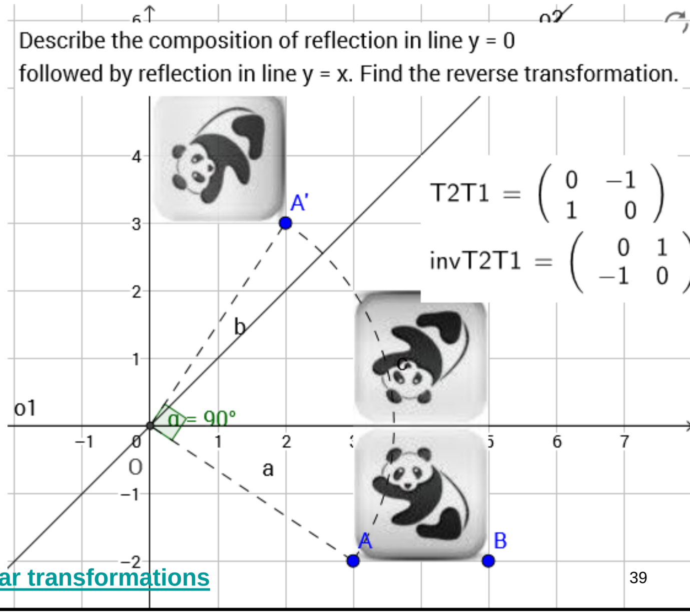
$$\text{Rot} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}; \quad \text{Ref} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\text{Rot*Ref} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}$$

$$\text{Ref*Rot} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & -\cos\alpha \end{pmatrix}$$



Composition of Linear Transformations



[Composition of linear transformations]

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*Decomposing Linear Transformations

- Any 2D Linear Transformation can be decomposed into the product of a rotation, a scale (or line reflection), and a rotation $M = R_1SR_2$.
- Any 2D congruence can be decomposed into the product of 3 line reflection at the most.

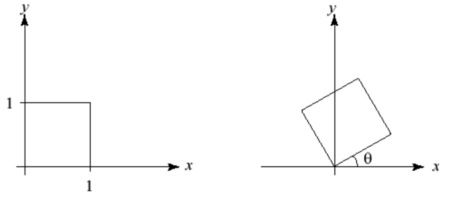
Isometry (congruent transformation)

- *Isometry* preserves length, whereas *direct isometry* preserves orientation and *opposite* does not preserve orientation
- Direct Isometry |R| = 1 (Rotation)
- Opposite Isometry |R| = -1 (Line Reflection)

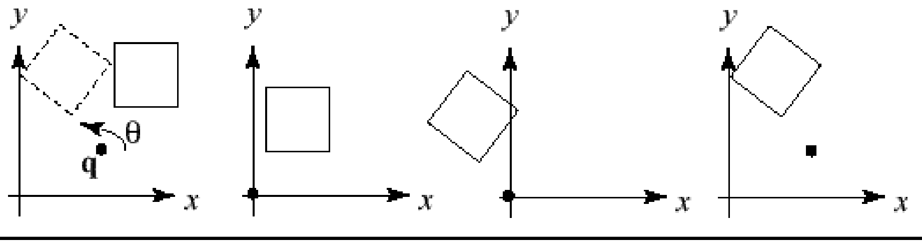
Linear Transformations

- Scale, Reflection, Rotation, and Shear are all linear transformations
- They satisfy: T(*a***u** + *b***v**) = *a*T(**u**) + *b*T(**v**)
  - **u** and **v** are vectors
  - *a* and *b* are scalars
- If T is a linear transformation
  - T((0, 0)) = (0, 0)
- **What important operation does that leave out?**

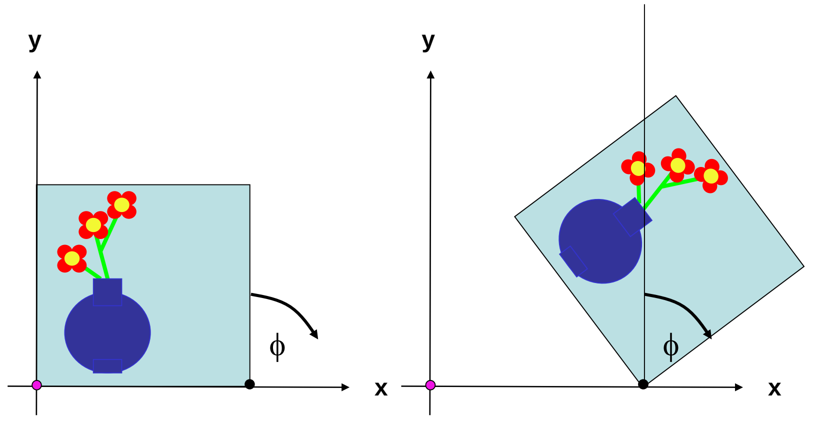
Linear transformation



Affine transformation



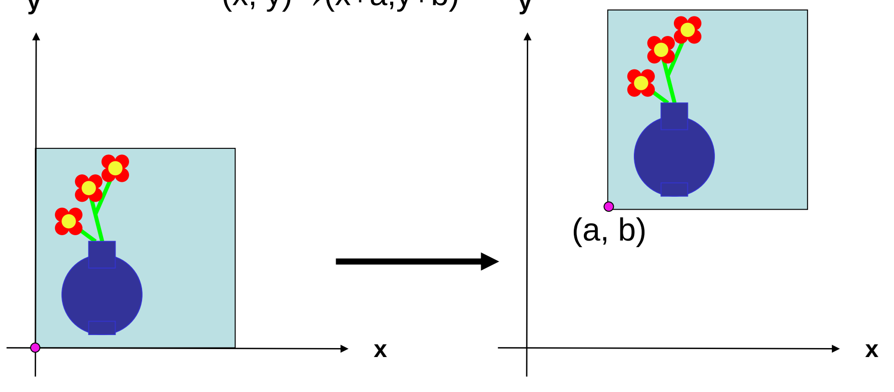
Rotation about an Arbitrary Point



This is not a linear transformation. The origin moves.

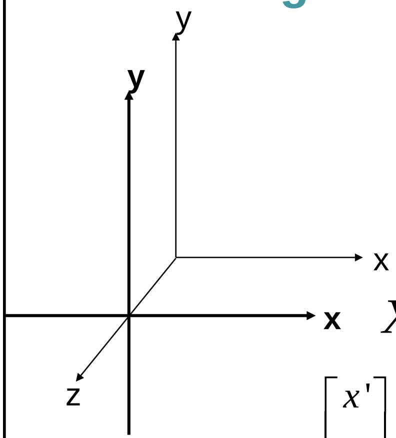
Translation

$(x, y) \rightarrow (x+a, y+b)$



This is not a linear transformation. The origin moves.

Homogeneous Coordinates



Embed the xy-plane in R^3 at z = 1.

(x, y) ↔ (x, y, 1)

$$X' = A \cdot X$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \\ 1 \end{bmatrix}$$

2D Linear Transformations as 3D Matrices

Any 2D linear transformation can be represented by a 2x2 matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{bmatrix}$$

or a 3x3 matrix

$$\begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \\ 1 \end{bmatrix}$$

2D afinne transformation

Image of a **point** $(x, y, 1)^T$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} e_1 & f_1 & m \\ e_2 & f_2 & n \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} e_1 x + f_1 y + m \\ e_2 x + f_2 y + n \\ 1 \end{bmatrix}$$

Image of a **vector** $(x, y, 0)^T$

$$\begin{bmatrix} x' \\ y' \\ 0 \end{bmatrix} = \begin{bmatrix} e_1 & f_1 & m \\ e_2 & f_2 & n \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} e_1 x + f_1 y \\ e_2 x + f_2 y \\ 0 \end{bmatrix}$$

2D afinne transformation

Image of a **origin** $(0, 0, 1)^T$ is 3rd column.

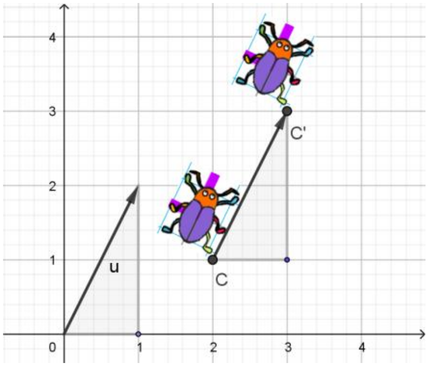
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} e_1 & f_1 & m \\ e_2 & f_2 & n \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} m \\ n \\ 1 \end{bmatrix}$$

Image of a **basis vectors** $(1, 0, 0)^T$, $(0, 1, 0)^T$ are 1st and 3rd columns.

$$\begin{bmatrix} e_1 & f_1 & m \\ e_2 & f_2 & n \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} e_1 & f_1 & m \\ e_2 & f_2 & n \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ 0 \end{bmatrix}$$

2D Linear Translations as 3D Matrices

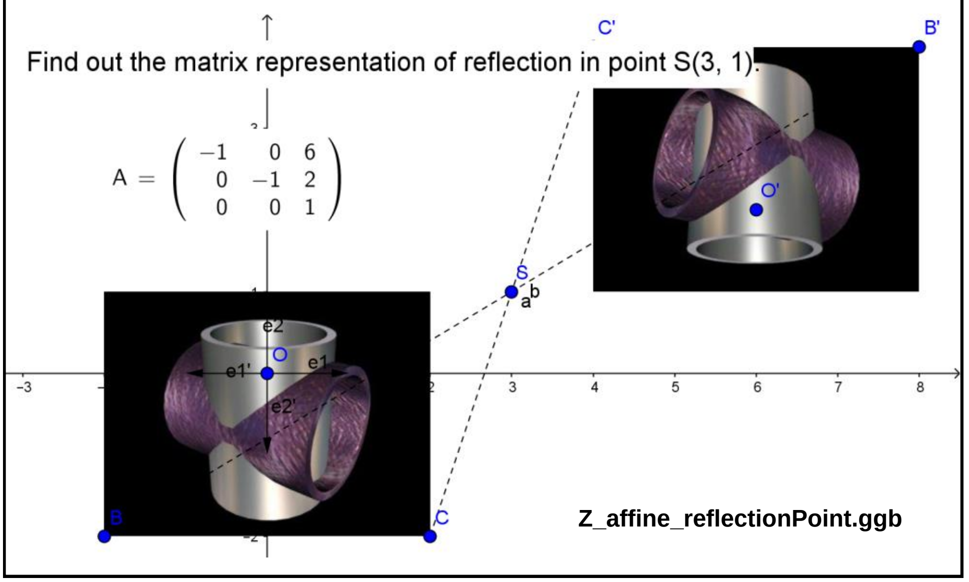
Any 2D translation can be represented by a 3x3 matrix.



$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \\ 1 \end{bmatrix}$$

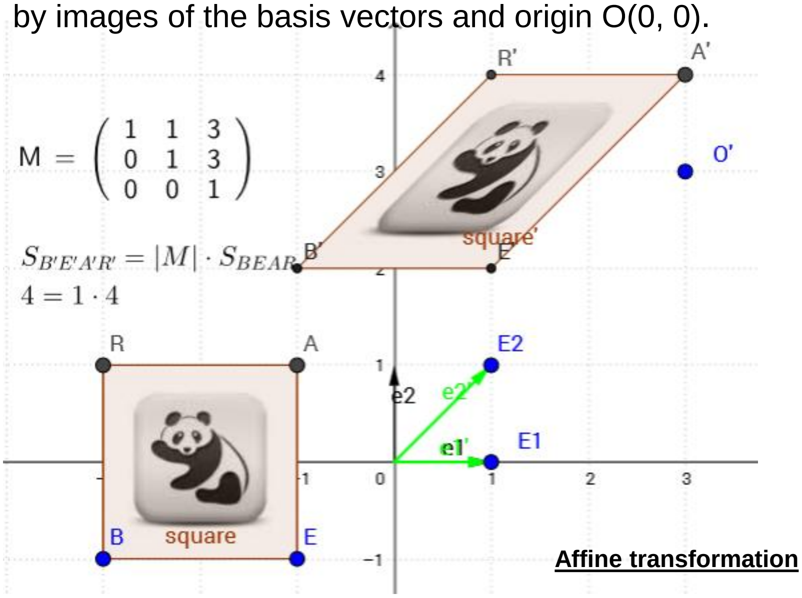
With homogeneous coordinates, we can represent all 2D affine transformations as 3D linear transformations.
We can then use matrix multiplication to transform objects.

Rotation 180° about an arbitrary point



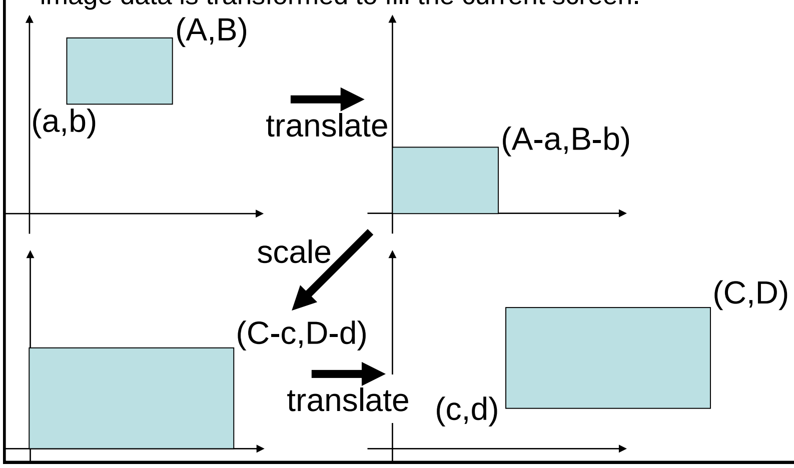
Z_affine_reflectionPoint.ggb

Recall that the column vectors of the matrix *M* are given by images of the basis vectors and origin O(0, 0).



Windowing Transforms

Windowing is the process of transforming co-ordinates from one space to another. It is used when scaling and transforming the view of a program. For example: when you zoom into an image, the original image data is transformed to fill the current screen.



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Fixed point of the plane isometry

Classify the transformation A. Determine all fixed points and directions.

$$A=\begin{pmatrix}0 & -1 & 6\\ 1 & 0 & 1\\ 0 & 0 & 1\end{pmatrix}\qquad \begin{aligned}\vec{v}' &= \lambda\vec{v}\\ \lambda\vec{v} &= \bar{A}\vec{v}\\ \left(\bar{A}-\lambda E\right)\vec{v} &= \vec{o}\end{aligned}\qquad \bar{A}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

Determinant A = 1, first two orthonormal columns yields the congruent transformation. Vectors could be investigated by linear part $\bar{A}$ of matrix A. Translation has no influence on vectors.

$$\left|\bar{A}-\lambda E\right| = \lambda^2+1=0$$

Characteristic polynomial has only complex solution. Isometry without fixed direction is **rotation**.

Fixed point of the plane isometry

Classify the transformation A. Determine all fixed points.

$$A=\begin{pmatrix}0 & -1 & 6\\ 1 & 0 & 1\\ 0 & 0 & 1\end{pmatrix}\qquad \begin{aligned}X' &= X\\ X &= AX\\ \left(A-E\right)X &= \vec{o}\end{aligned}$$

$$\left(A-E\right)=\begin{pmatrix}-1 & -1 & 6\\ 1 & -1 & 1\\ 0 & 0 & 0\end{pmatrix}\sim\begin{pmatrix}-1 & -1 & 6\\ 0 & -2 & 7\\ 0 & 0 & 0\end{pmatrix}\sim\overset{*}{\boxed{\begin{pmatrix}1 & 0 & -2.5\\ 0 & 1 & -3.5\\ 0 & 0 & 0\end{pmatrix}}}$$

GeoGebra tool `ReducedRowEchelonForm(M)` provides Gaussian elimination with echelon form *. Using back-substitution, unknowns x, y can be solved for. Transformation has only one fixed point FP = (2.5, 3.5).

Fixed point of the plane isometry

Classify the transformation A. Determine all fixed points.

$$\begin{pmatrix} 1 & 0 & -2.5 \\ 0 & 1 & -3.5 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x - 2.5 = 0$$

$$y - 3.5 = 0$$

GeoGebra tool `ReducedRowEchelonForm(M)` provides Gaussian elimination with echelon form *. Using back-substitution, unknowns x, y can be solved for.
Transformation has only one fixed point FP = (2.5, 3.5).

3D Transformations

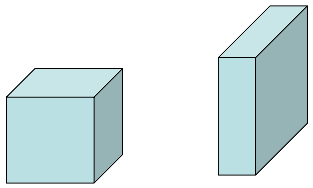
Remember:
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \leftrightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

A 3D linear transformation can be represented by a 3x3 matrix.

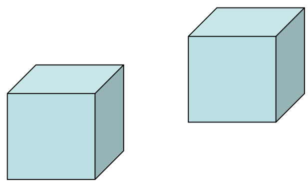
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \leftrightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3D Affine Transformations

$$\text{scale}(s_x, s_y, s_z) = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

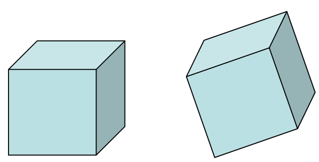


$$\text{translate}(t_x, t_y, t_z) = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



3D Rotations

$$\text{rotate}_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$\text{rotate}_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{rotate}_z(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$