Functions and their representations

- Function arise whenever one quantity depends on another

The area A of a circle depends on the radius r of the circle. The rule that connects r and A is given by the equation $A = \pi r^2$. With each positive number r there is associated one value of A, and we say that A is a *function* of r.

The human population of the world P depends on the time t. The table gives estimates of the world population $P(t)$ at time t, for certain years. For instance,

$$P(1950) \approx 2{,}560{,}000{,}000$$

But for each value of the time t there is a corresponding value of P, and we say that P is a function of t.

Year	Population (millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080
2010	6870

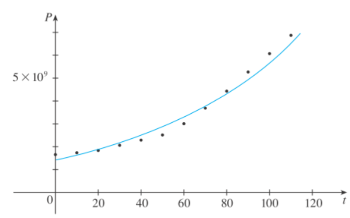
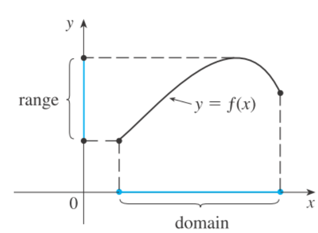


FIGURE 8 Graph of a mathematical model for population growth

Functions and their representations

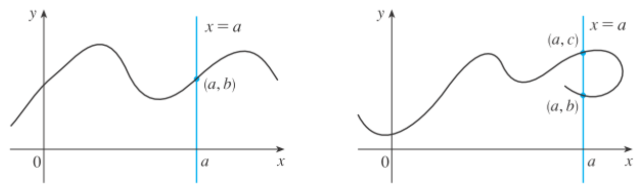
> A **function** f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E.



The most common method for visualizing a function is its graph. If f is a function with domain D, then its **graph** is the set of ordered pairs

$$\{(x, f(x)) \mid x \in D\}$$

Functions and their representations



Vertical test: A curve is the graph of a function of x if and only if no vertical line intersects the curve more than ones.

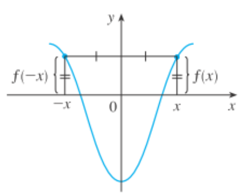


FIGURE 14 An even function

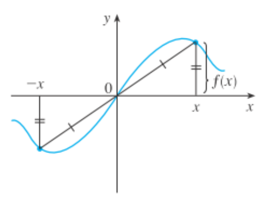
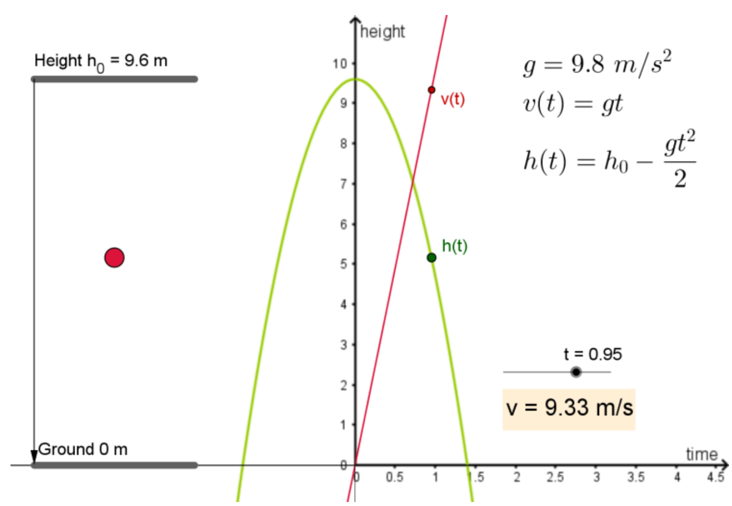


FIGURE 15 An odd function

Ex: Free fall in Newtonian mechanics

- **Uniform gravitational field without air resistance**
- Constant acceleration g.

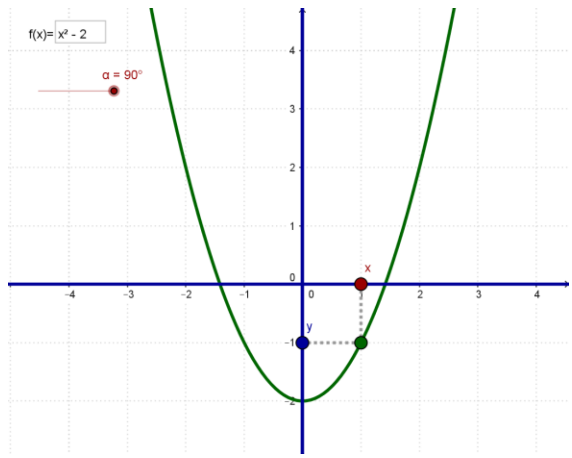


Function of one variable

- Graph the fuction, find all important properties

- D – domain
- H – range
- Injective function
- Increasing/decreasing

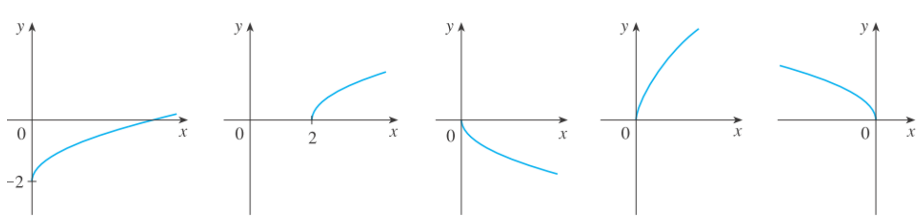
  $x_1 < x_2 \Rightarrow y_1 < y_2$
- Extrema
- Inflection points



Affine Transformation of Functions

By applying certain transformations to the graph of a given function we can obtain the graphs of certain related functions. This will give us the ability to sketch the graphs of many functions quickly by hand. It will also enable us to write equations for given graphs.

A) $\mathrm{y} = \sqrt{-x}$ B) $\mathrm{y} = \sqrt{x} - 2$ C) $\mathrm{y} = -\sqrt{x}$ D) $\mathrm{y} = 2\sqrt{x}$ E) $\mathrm{y} = \sqrt{x - 2}$



Translation

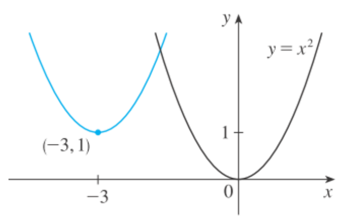
VERTICAL AND HORIZONTAL SHIFTS Suppose $c > 0$. To obtain the graph of

$y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward

$y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward

$y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right

$y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left



Scaling
Horizontal and vertical reflection

VERTICAL AND HORIZONTAL STRETCHING, SHRINKING, AND REFLECTING
Suppose $c > 1$. To obtain the graph of

$y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c

$y = (1/c)f(x)$, shrink the graph of $y = f(x)$ vertically by a factor of c

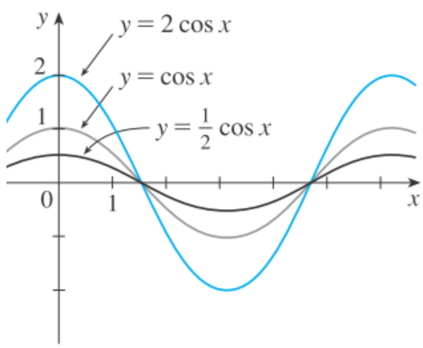
$y = f(cx)$, shrink the graph of $y = f(x)$ horizontally by a factor of c

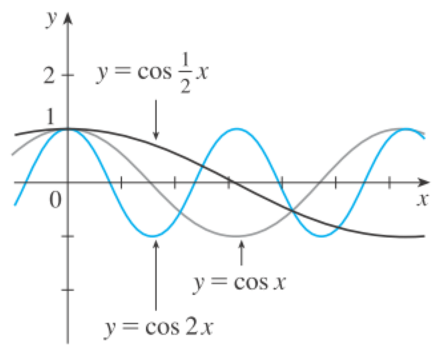
$y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c

$y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis

$y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis

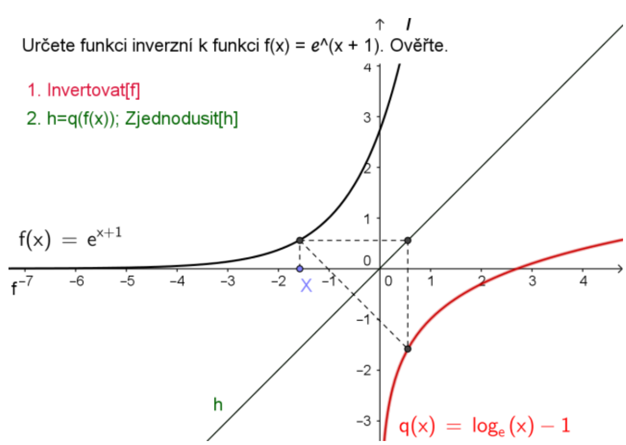
Scaling
Horizontal and vertical reflection





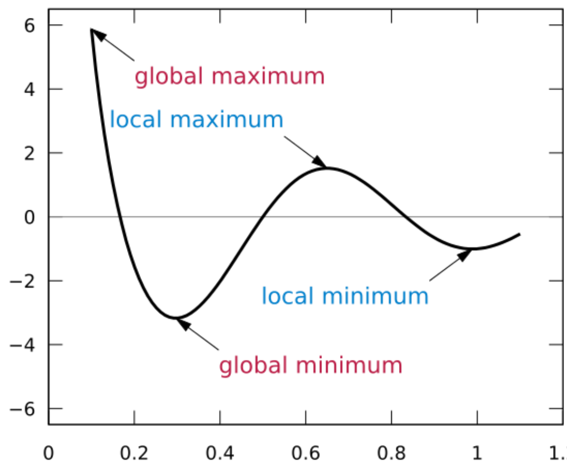
Inverse function

Inverse function is a function that "reverses" another function: if the function *f* applied to an input *x* gives a result of *y*, then applying its inverse function *g* to *y* gives the result *x*, and vice versa. I.e., *f*(*x*) = *y* if and only if *g*(*y*) = *x*.
Composition of two mutually inverse functions gives identity function y = x.



Maxima and minima (turning points)

The **maxima and minima** (the respective plurals of **maximum** and **minimum**) of a function, known collectively as **extrema** (the plural of **extremum**), are the largest and smallest value of the function



21–24 ▪ Evaluate the difference quotient for the given function. Simplify your answer.

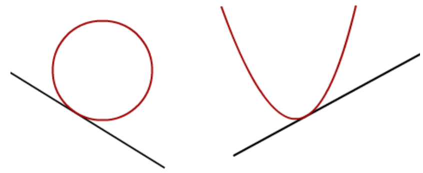
21. $f(x) = 4 + 3x - x^2, \quad \dfrac{f(3+h) - f(3)}{h}$

22. $f(x) = x^3, \quad \dfrac{f(a+h) - f(a)}{h}$

23. $f(x) = \dfrac{1}{x}, \quad \dfrac{f(x) - f(a)}{x - a}$

24. $f(x) = \dfrac{x+3}{x+1}, \quad \dfrac{f(x) - f(1)}{x - 1}$

Tangent line



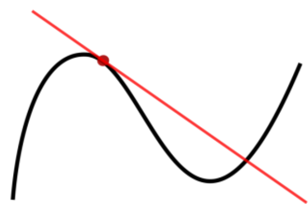
The tangent line to a plane curve at a given point is the straight line that "just touches" the curve at that point.

Euclid *Elements* (c. 300 BC)

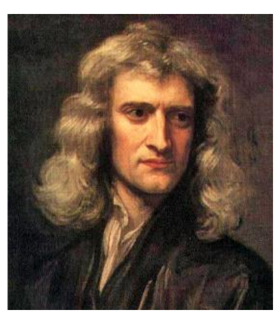
Apollonius : *Conics* (c. 225 BC) " *line such that no other straight line could fall between it and the curve"*

Fermat (1630) *technique of adequality* similir to taking |(x+h) - f(x)|/h

Leibniz Leibniz (1684) defined it as the line through a pair of infinitely close points on the curve.



Isaac Newton 1643-1727
Gottfried Wilhelm Leibniz 1646-1716

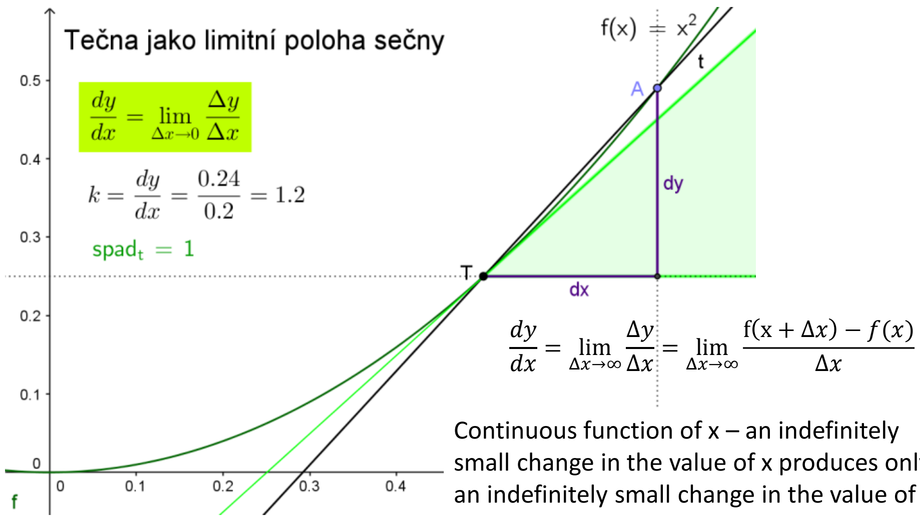


Perhaps one the most infamous controversies in the history of science.

I. Newton: "the method of fluxions and fluents") in 1666, at the age of 23, but did not publish it
II. G. W. Leibniz "Nova Methodus pro Maximis et Minimis". In 1684
III. Newton's *Principia* of 1687 was "nearly all about this calculus").

Derivative as a slope of curve

Leibniz's notation for infinitesimal small quantities *dy*, *dx*.

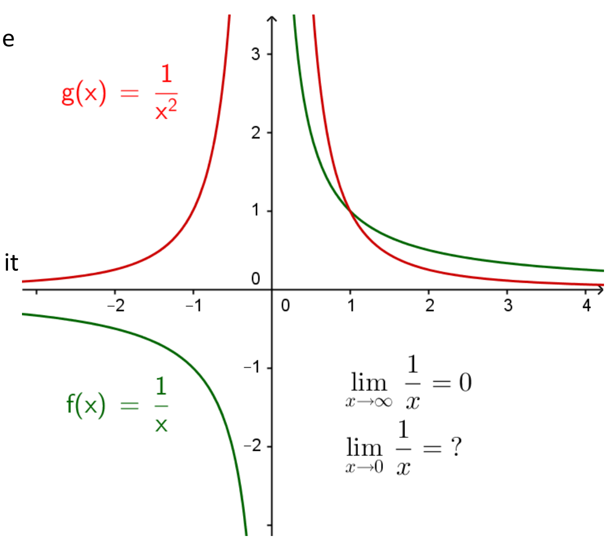


Continuous function of x – an indefinitely small change in the value of x produces only an indefinitely small change in the value of y.

Limit of a function

Bolzano , 1817, introduced the basics of the epsilon-delta technique to define continuous functions.

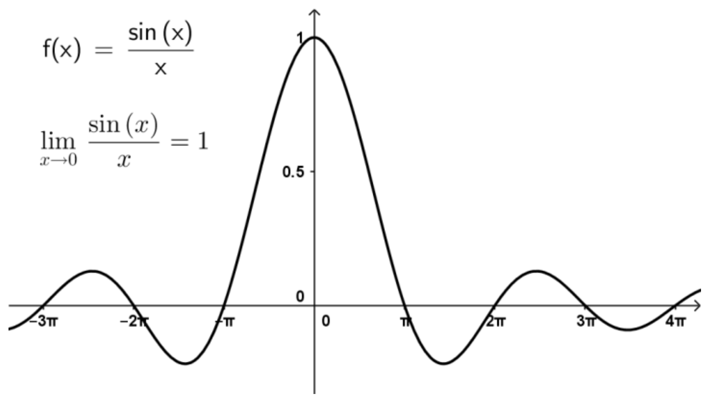
Weierstrass 1820 first introduced the epsilon-delta definition of limit in the form it is usually written today. He also introduced the notation



Limit of a function

Let's investigate the behavior of the function f(x) for a values x near 0.

From the graph we see that when x is close to 0 (on either side of 0), f(x) is close to 1.



Limit of a function

$\boxed{1}$ **DEFINITION** Suppose $f(x)$ is defined when x is near the number a. (This means that f is defined on some open interval that contains a, except possibly at a itself.) Then we write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

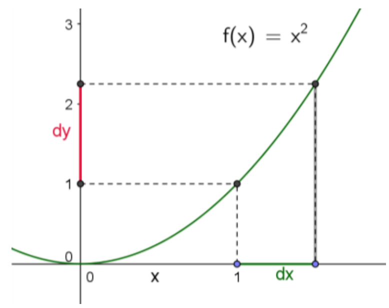
Leibniz: Calculus differentialis

Leibniz set up 3 main rules for infinitesimals in work. „Calculus differentialis" .
Ex. Derivative of parabola f(x) = y = x^2

$$\frac{dy}{dx} = \frac{(\mathrm{x} + \mathrm{dx})^2 - x^2}{\mathrm{dx}} = \frac{x^2 + 2xdx + dx^2 - x^2}{dx} = 2x + dx$$

$dy = 2xdx + dx^2$; but dx^2 tend to zero

$\mathrm{dy} = 2xdx$



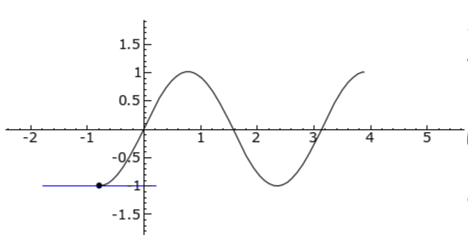
Limit notation

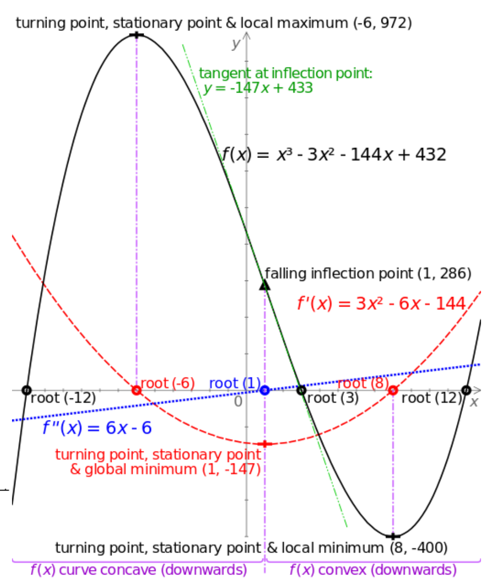
$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\mathrm{f}(\mathrm{x} + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\mathrm{x}\Delta x + \Delta x^2}{\Delta x} = \lim_{\Delta x \to 0} 2\mathrm{x} + \Delta x = 2x$$

Inflection point

The **inflection point** is a point on a curve at which the curve changes from being concave (concave downward) to convex (concave upward), or vice versa.

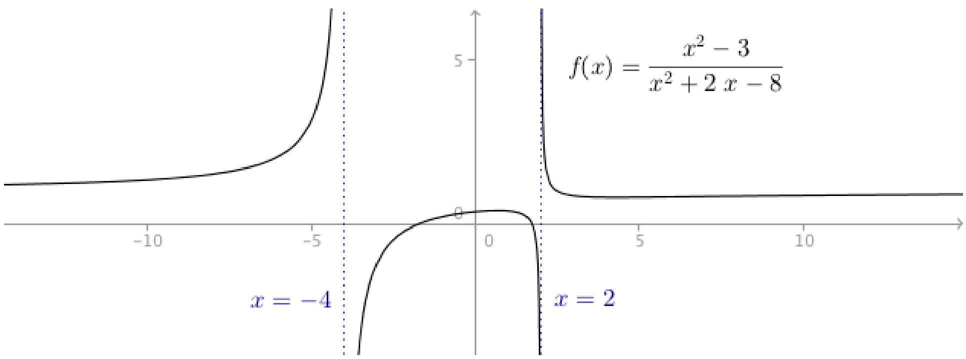
Inflection points are the points of the curve where the curvature changes its sign while a tangent exists





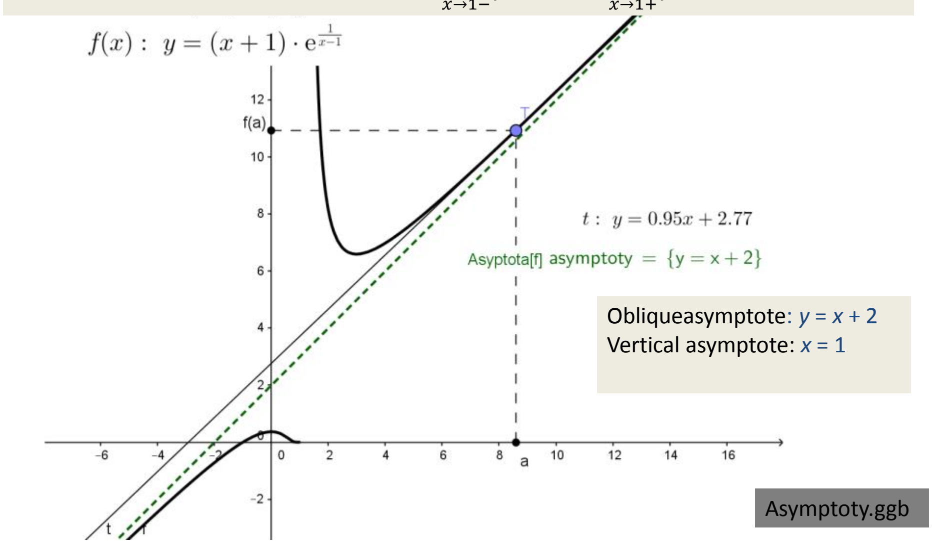
Asymptote

A line that a curve approaches, as it heads towards infinity.
The **distance** between the curve and the asymptote **tends to zero** as they head to infinity (or −infinity).
For oblique asymtpte y = kx+q
$k = \lim_{x\to\infty} \frac{f(x)}{x}$, q $= \lim_{x\to\infty} (f(x) - k(x))$



Ex: Asymptote

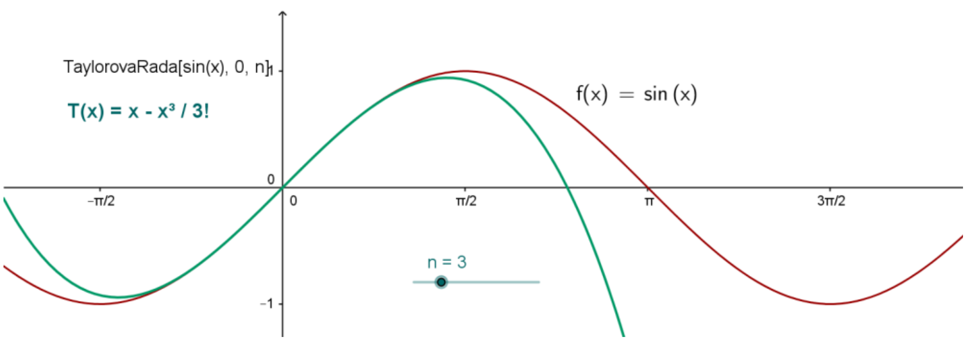
Domain D(f) = R-{1}, discontunuity for x = 1 $\lim_{x\to 1-} f(x) = 0$ a $\lim_{x\to 1+} f(x) = \infty$.



Taylor polynomial approximation

V případě existence všech konečných derivací funkce f v bodě a
lze Taylorovu řadu zapsat jako

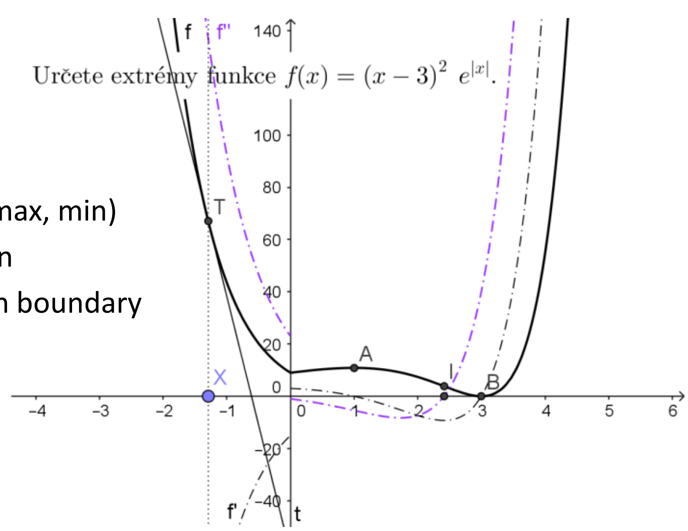
$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \ldots = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$



Investigate the graph of the function

Určete extrémy funkce $f(x) = (x-3)^2\, e^{|x|}$.

- Domain
- Range
- Turning points (max, min)
- Poin of inflection
- Limits on domain boundary
- Asymptote



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