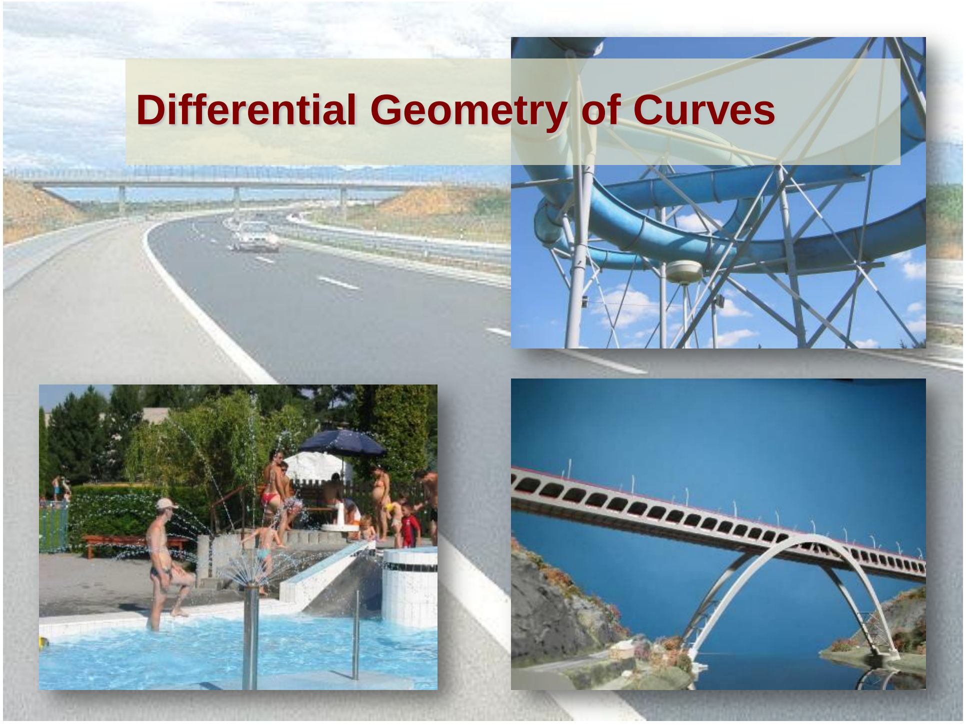
Differential Geometry of Curves

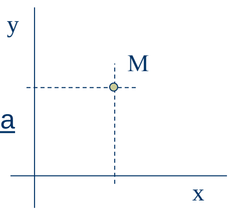


Cartesian coordinate system

René Descartes (1596-1650) (lat. Renatus Cartesius)

French philosopher, mathematician, and scientist.

- *Rationalism*
- *„Ego cogito, ergo sum* (**I think, therefore I am)**
- *Father of analytic geometry*
- long correspondence with Princess Elisabeth of Bohemia *(1618-1680)* devoted mainly to moral and psychological subjects.
- *1649* Queen Christina of Sweden invited Descartes to her court to tutor her in his ideas about love *(1626-1689)*



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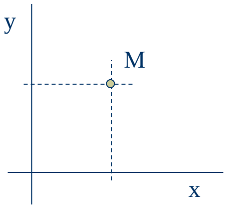
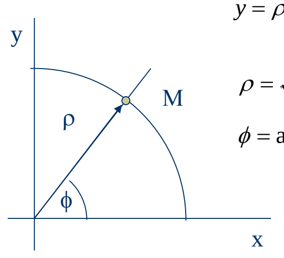
Polar coordinate system

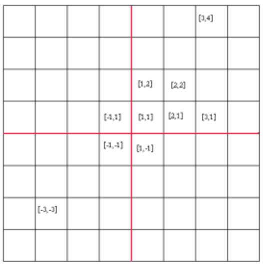
- Cartesian coordinates

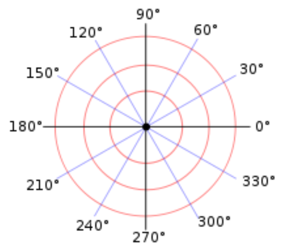
Polar coordinates.

$$x = \rho\cos\phi$$
$$y = \rho\sin\phi,\ |\rho| \geq 0$$
$$\rho = \sqrt{x^2 + y^2}$$
$$\phi = \arctan\frac{y}{x}$$





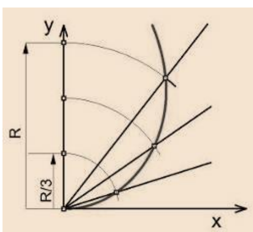




Spirals

- Archimedova spirála

$$\rho = a\phi$$

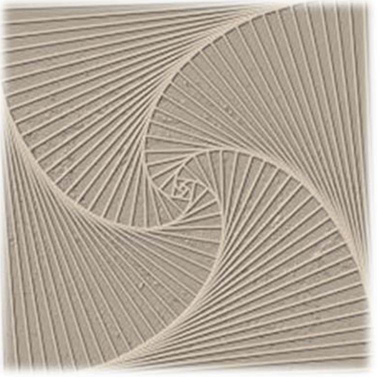


- Logaritmická spirála

$$\rho = a \cdot \mathrm{e}^{(b\phi)}$$

Spira mirabilis

Jacob Bernoulli 1654-1705) requested that the curve be engraved upon his tomb with the phrase "Eadem mutata resurgo".



I shall arise the same, though changed

Representation of a curve

- Parametric (vector form) $X(t) = [x(t), y(t)]$
- Implicit equation $F(x, y) = 0$, *e.g.* $\frac{x^2}{4} + \frac{y^2}{9} = 1$
- Graph of a function $y = f(x)$, e. g. $y = (x-3)^2 e^{|x|}$

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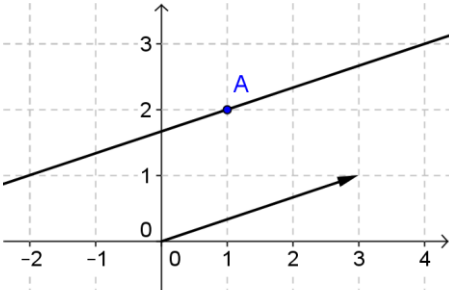
Example:

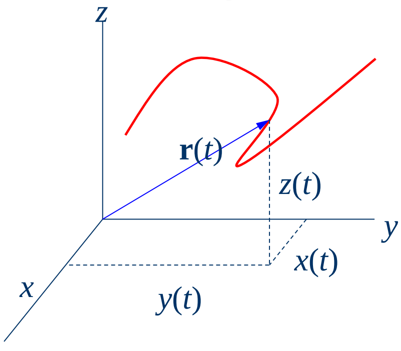
$$X(t) = [1 + 3t,\ 2 + t]$$

$$x - 3y + 5 = 0$$

$$y = \frac{1}{3}x + \frac{5}{3}$$



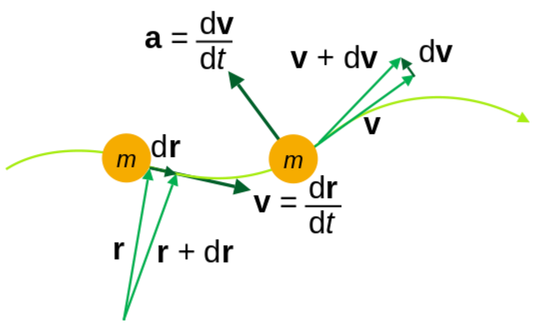
Equation of Motion, Position Vector



Position vector defines the motion of a particle (i.e. a point mass) – its location relative to a given coordinate system at some time t

$$\mathbf{r}(t) = [x(t), y(t), z(t)]$$

Kinematic quantities of a classical particle: mass *m*, position **r**, velocity **v**, acceleration **a**

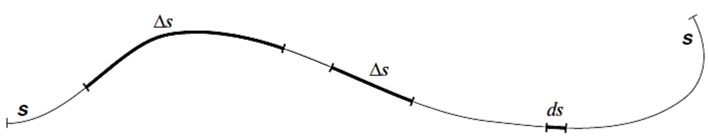


Velocity

The velocity of an object is the **rate of change** of its position with respect to a frame of reference, and is a function of time. Velocity is equivalent to a specification of its speed and direction of motion (e.g. 60 km/h to the north).

Average velocity.

$$v = \frac{\Delta s}{\Delta t};\ \Delta t = t_2 - t_1;\ \Delta s = s(t_2) - s(t_1)$$



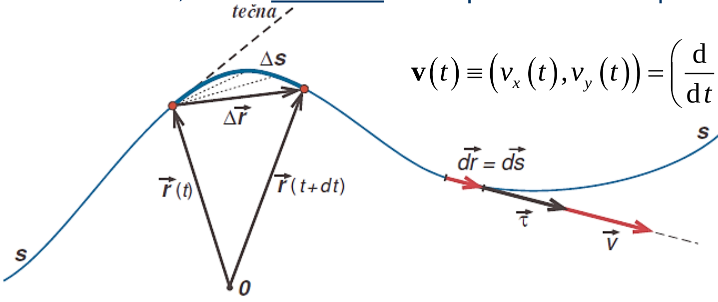
Instantaneous velocity

- the rate of change of position with respect to time
- Speed – the magnitude of the velocity

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{s(t+\Delta t) - s(t)}{\Delta t} = \frac{ds}{dt}$$

- If we consider ***v*** as velocity and ***r*** as the position vector, then we can express the (instantaneous) velocity of a particle or object, at any particular time *t*, as the derivative of the position with respect to time.

$$\mathbf{v}(t) \equiv \left(v_x(t), v_y(t)\right) = \left(\frac{\mathrm{d}}{\mathrm{d}t}x(t), \frac{\mathrm{d}}{\mathrm{d}t}y(t)\right)$$



Free fall

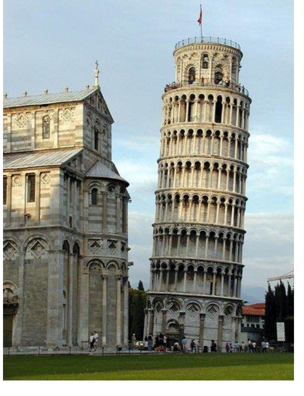
the rate of change of position with respect to time

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{s(t+\Delta t) - s(t)}{\Delta t} = \frac{ds}{dt}$$

Example: Free fall
The absence of an air resistance

$$s = \frac{1}{2}gt^2$$

$$v = \frac{ds}{dt} = gt$$



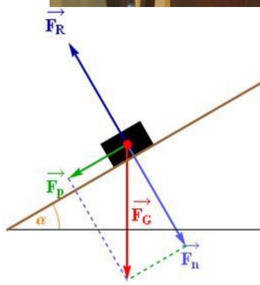
Heavy objects fall faster than lighter ones, in direct proportion to weight.
Aristotélēs (384 – 322)

Inclined plane

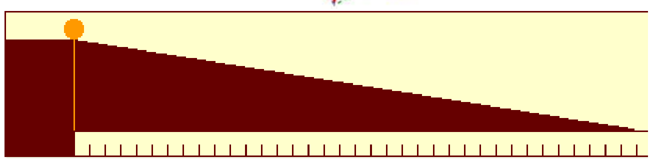
Traveled distance is proportional to the square of the time.

$$s = \frac{1}{2}at^2 = \frac{1}{2}gt^2 \sin\alpha$$

The final velocity is dependent only on the height and not the angle.



t	s	Δs
1	1	
2	4	3
3	9	5
4	16	7
5	25	9
6	36	11

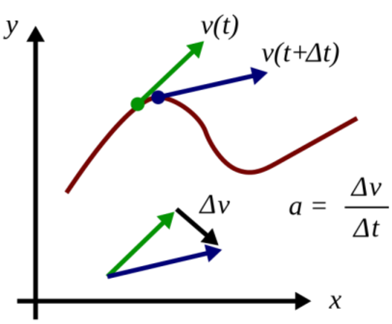


Acceleration
the rate of change of velocity of an object with respect to time

$$d\vec{v} = \lim_{\Delta t \to 0} \Delta \vec{v}$$

$$d\vec{v} = \vec{v}(t+dt) - \vec{v}(t)$$

Acceleration is NOT necessary on a normal line

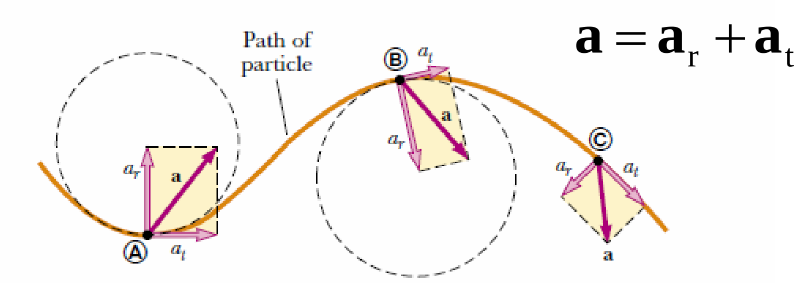


$$\vec{a} = \frac{d\vec{v}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t}$$

$$\mathbf{a}(t) = \lim_{\Delta t \to 0} \frac{\mathbf{v}(t+\Delta t) - \mathbf{v}(t)}{\Delta t} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(t)$$

$$\mathbf{a}(t) \equiv \left(a_x(t), a_y(t)\right) = \left(\frac{\mathrm{d}}{\mathrm{d}t}v_x(t), \frac{\mathrm{d}}{\mathrm{d}t}v_y(t)\right)$$

Tangential and normal acceleration



$$\mathbf{a} = \mathbf{a}_r + \mathbf{a}_t$$

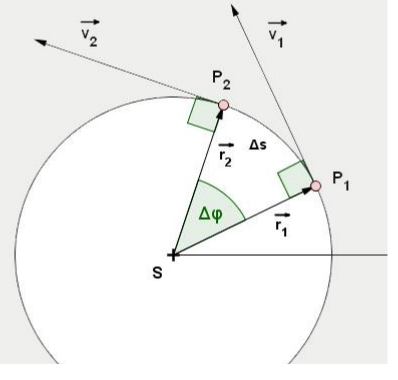
The tangential component **a**t *is due to the change in speed of traversal, and points along the curve in the direction of the velocity vector (or in the opposite direction).*
*The normal component **ar*** is due to the change in direction of the velocity vector and is normal to the trajectory, pointing toward the center of curvature of the path.

Uniform circular motion

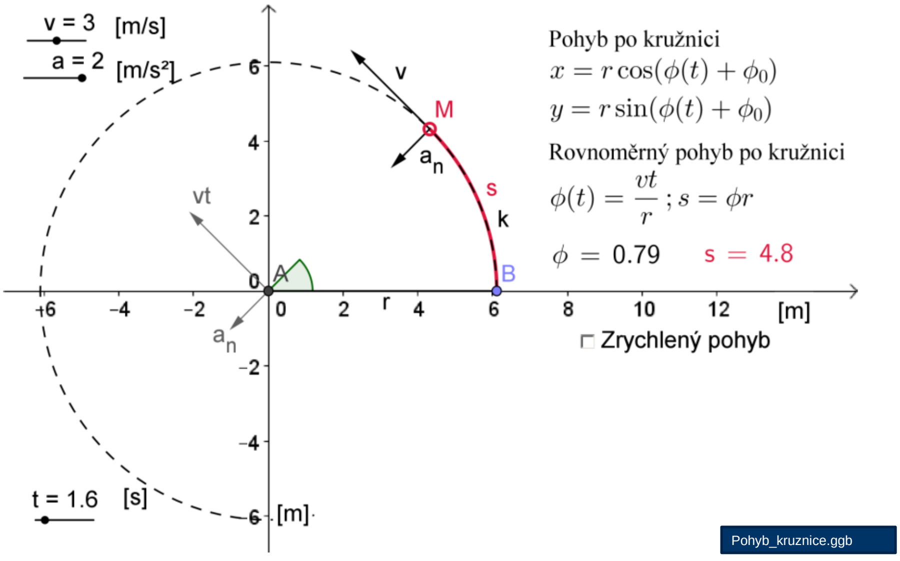
$$\boxed{s = \left[r.\cos\varphi(t), r.\sin\varphi(t)\right]}$$

- Instantaneous velocity $v = \dfrac{ds}{dt} = \left[-\dfrac{d\varphi}{dt} r\sin\varphi(t); \dfrac{d\varphi}{dt} r\cos\varphi(t)\right]; |v| = r\dfrac{d\varphi}{dt}$
- Angular rate $\omega = \dfrac{d\varphi}{dt} = \dfrac{|v|}{r}$

$$\boxed{\varphi(t) = \int \frac{|v|}{r} dt = \frac{|v|}{r} t}$$



Circular motion



Simple gravity pendulum

The period of swing of a simple gravity pendulum depends on its length and the local strength of gravity.

$$\varphi''(t) = -\frac{g}{l}\varphi(t)$$

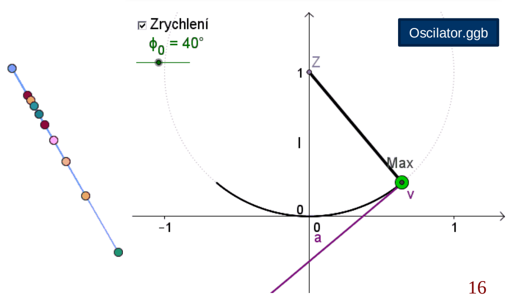
$$\textit{assume} \;\; \sin\varphi(t) \approx \varphi(t)$$

$$\varphi(t) = \varphi_0 \cos\left(t\sqrt{\frac{g}{l}}\right)$$

$$X(t) = \left[l\sin\varphi(t); l - l\cos\varphi(t)\right]$$

The period is independent of amplitude

$$T = 2\pi\sqrt{\frac{l}{g}}$$



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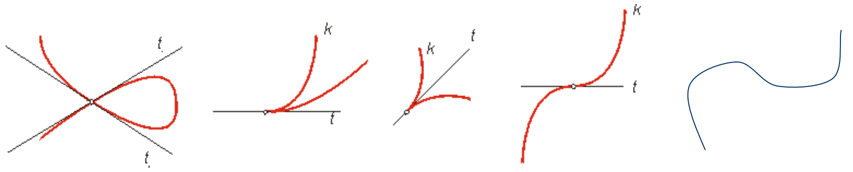
Vector function as parametrization of smooth curve C^n

The set $k \subset E^3$ of all points in space is called a space curve C^n if point coordinates could be expressed by mapping $I \to R^3$, $t \to X(t)$, where parameter t varies throughout the interval I and

- $X(t)$ is continuous on an interval I
- $X(t)$ is injective (one to one)
- $X(t)$ has continuous n-th derivatives on a interval I
- $X(t)$ is regular - derivative vector $X'(t) \neq o$.



Curve in plane	$X(t) = [x(t); y(t)]$
Curve in space	$X(t) = [x(t); y(t); z(t)]$

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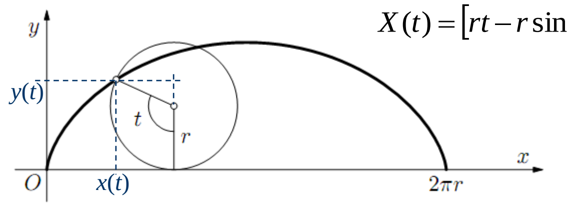
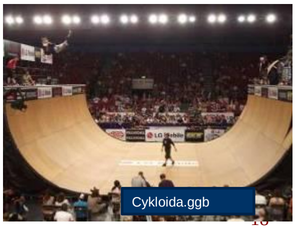
Cycloid

- A **cycloid** is the curve traced by a point on the perimeter of a circular wheel as the wheel rolls along a straight line without slipping.

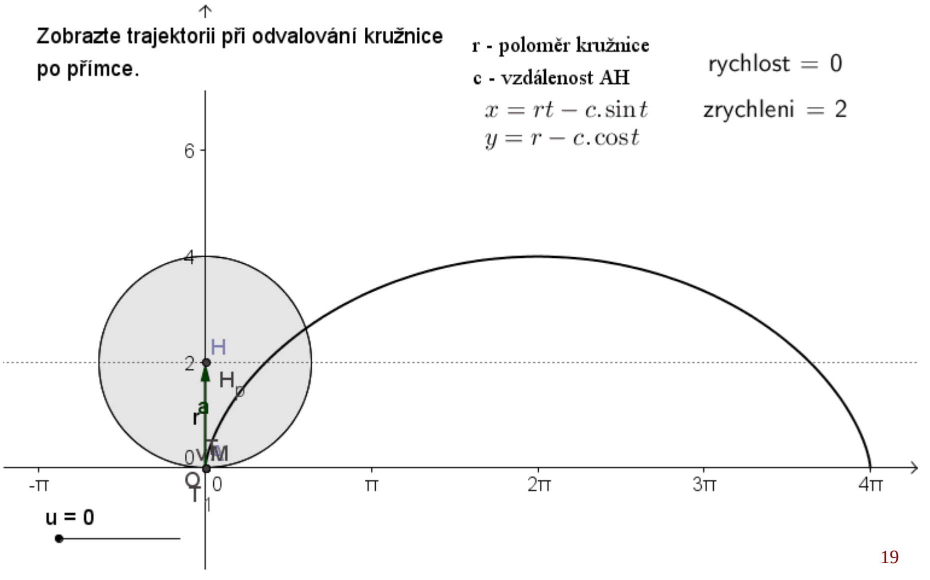
$$X(t) = [rt - r\sin t; r - r\cos t]$$





Cycloid

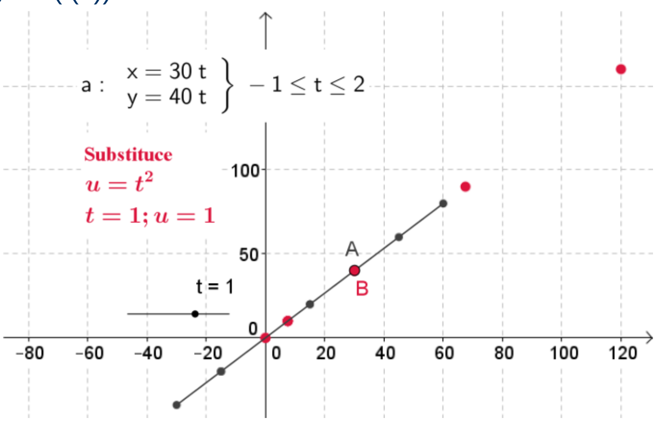
$$X(t) = [rt - r\sin t; r - r\cos t]$$



Reparametrization

Let X(t) $t \in I$ and Y(u), $u \in J$ denote parametrizations for a curve. They have the same image (and run through it in the same direction) if there exist a strictly increasing differentiable function t = f(u) such that

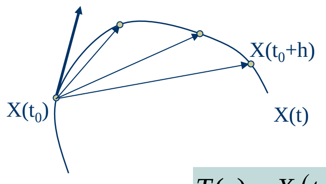
1. f(u) is one to one I $\leftrightarrow$ J
2. Y(u) = X(f(u))

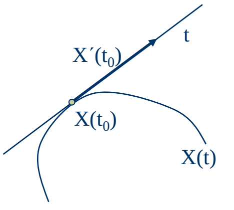


Tangent line

Curve X(*t*) has a tangent line at regular point X(t_0):

$$\lim_{h\to 0}\frac{X(t_0+h)-X(t_0)}{h}=X'(t_0)$$





$$T(r)=X(t_0)+r\cdot X'(t_0),\ r\in R$$

Tangent line is the limiting position of the secants connecting two points ont the curve close to the given one.

Ex: Tangent line to the curve y = f(x) at point f(x_0):

$$y=f(x)$$
$$X(t)=[t,f(t)]\Rightarrow X(t_0)=[t_0,f(t_0)]$$
$$X'(t)=[1,f'(t)]\Rightarrow X'(t_0)=[1,f'(t_0)]$$

$$x=t_0+r$$
$$y=f(t_0)+r\cdot f'(t_0)$$

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Tangent line

Curve X(*t*) has a tangent line at regular point X(t_0):

$$T(r)=X(t_0)+r\cdot X'(t_0),\ r\in R$$

Reparametrization give the derivative vector with the same direction.

$$K(t)=[\cos(t),\sin(t)];\qquad \frac{dK}{dt}=[-\sin(t),\cos(t)],$$

tangent vector at point K(0): $\frac{dK}{dt}(t=0)=[0,1]$

$$L(u)=[\cos(2u),\sin(2u)];\qquad \frac{dL}{du}=[-2\sin(2u),2\cos(2u)],$$

tangent vector at point L(0): $\frac{dL}{du}(u=0)=[0,2]$

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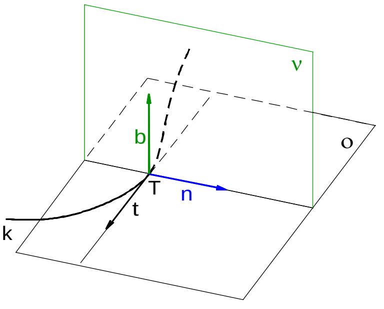
Frenet-Serret moving frame

- Vectors ***t, n, b*** form an orthonormal basis spanning ℝ3 and are defined as follows:

- Osculating plane –plane spanned by X' and X'', best approximating plane.

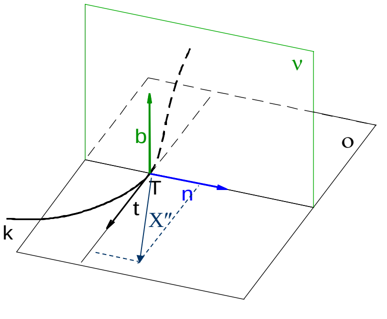
$$o \equiv X(t_0) + r \cdot X'(t_0) + s \cdot X''(t_0)\ ;\ r, s \in R$$



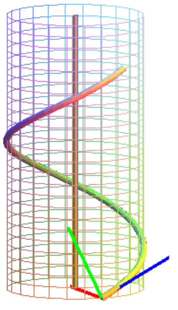
t – tangent
- *n* – normal
- *b* – binormal
- o = (*t*,*n*) – osculating plane
- ν = (*b*,*n*) – normal plane

Frenet-Serret moving frame

- Unit vector tangent to the curve pointing in the direction of motion.
  $$\vec{t} = \frac{\vec{X}'}{\left\|\vec{X}'\right\|}$$
- Binormal unit vector
  $$\vec{b} = \frac{\vec{X}' \times \vec{X}''}{\left\|\vec{X}' \times \vec{X}''\right\|}$$
- Normal unit vector
  $$\vec{n} = \vec{t} \times \vec{b}$$



Frenet-Serret moving frame of the helix

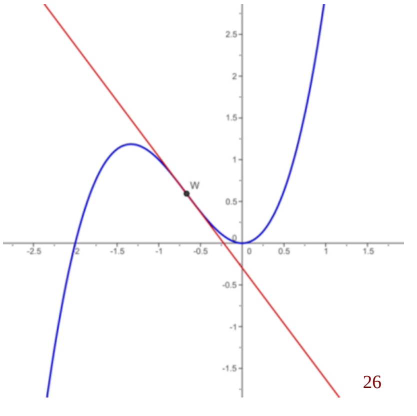


- tangent
- normal
- binormal

Point of inflection

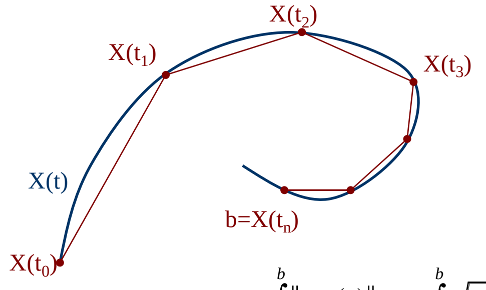
The Frenet–Serret formulas apply to curves which are non-degenerate, which roughly means that they have nonzero curvature. More formally, in this situation the velocity vector X′(t) and the acceleration vector X″(t) are required not to be proportional.

$$\vec{X}'(t_0) = \vec{X}''(t_0)$$



Arc length – rectification o a curve

Length of the curve X(*t*) between points X(*a*) a X(*b*)



Polygonal path

$$l=\sum_{i=0}^{n-1}\left\|X\left(t_{i+1}\right)-X\left(t_i\right)\right\|$$

$$l=\int_a^b\left\|X'(t)\right\|dt=\int_a^b\sqrt{X'(t)\cdot X'(t)}\,dt$$

We can then approximate the curve by a series of straight lines connecting the points.

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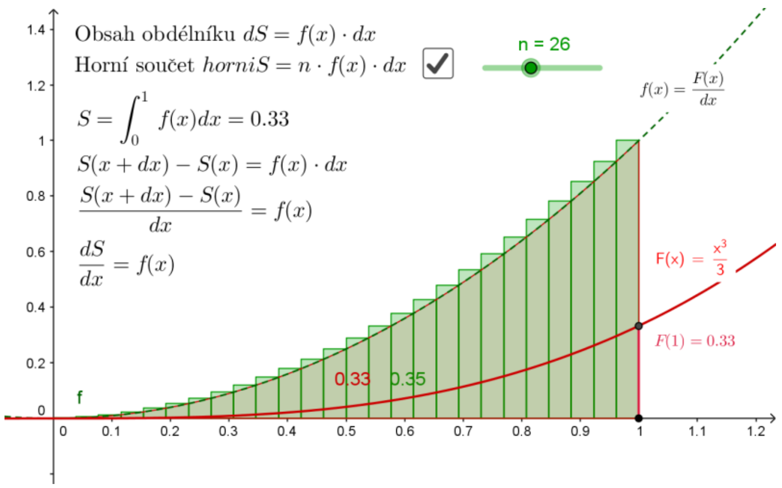
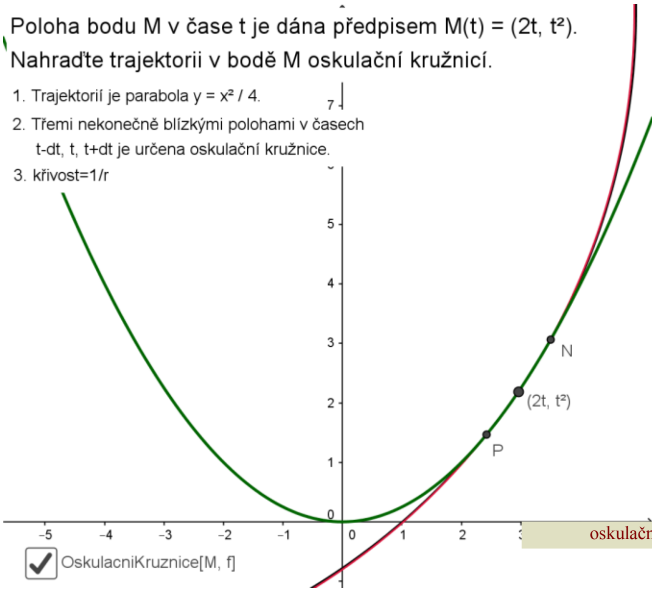
Integral

A definite integral of a function can be represented as the signed area of the region bounded by its graph.



Approximating circles and osculating circle



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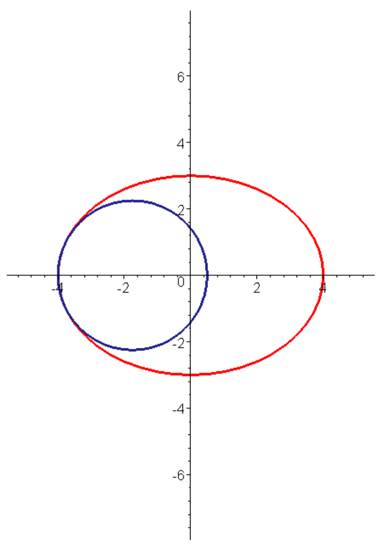
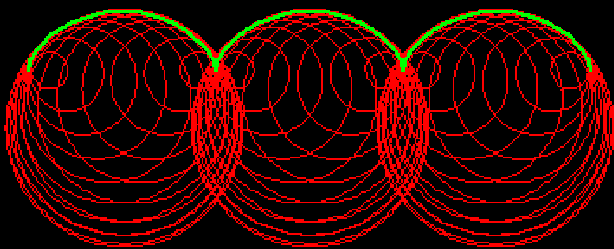
Osculating circle of the ellipse



Osculating circle of the cycloid



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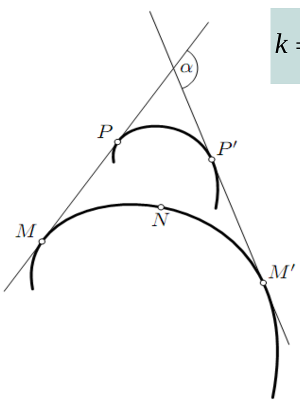
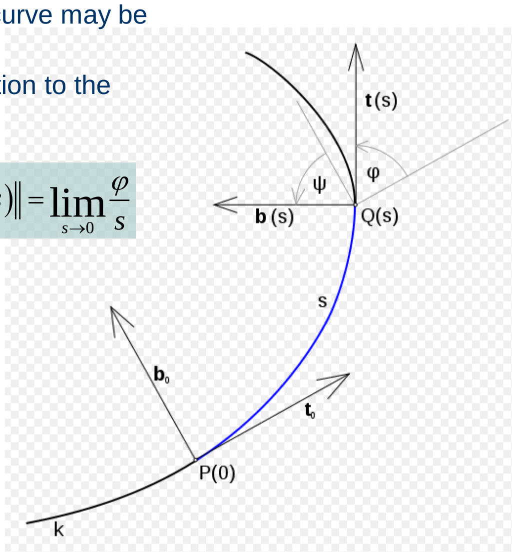
Curvature

Near a point P on the curve, that curve may be approximated by osculating circle.

The curvature is in inverse proportion to the radius of this osculating circle.

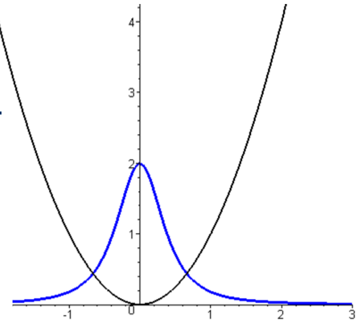


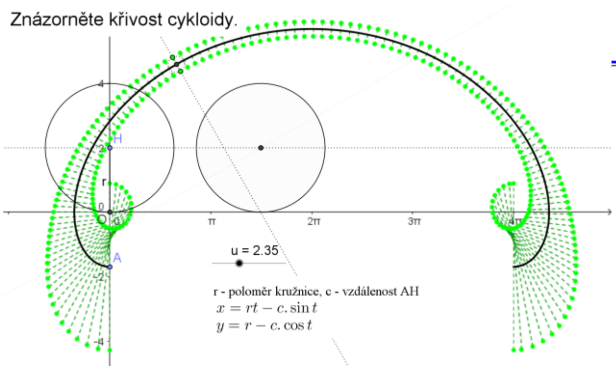
$$k = \|X''(s)\| = \lim_{s \to 0} \frac{\varphi}{s}$$



Geometric meaning of the curvature

- Circles and lines have constant curvature.
- Vertices of the curve have extremal curvature.
- Point of inflection has zero curvature.





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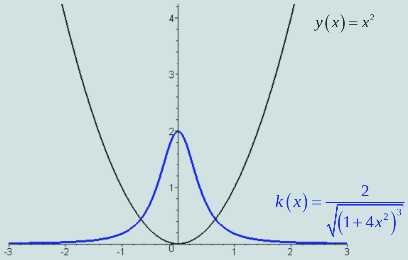
Curvature Calculator

1. X(*l*) parametrized by arc length $k = \|X''(l)\|$
2. X(*t*) parametrized by general parameter $k = \dfrac{\|X' \times X''\|}{\sqrt{|X' \cdot X'|^3}}$
3. Curve as a graph of a function *y* = f(*x*) $k = \dfrac{|y''|}{\sqrt{\left(1+(y')^2\right)^3}}$



Př: Vypočítejte funkci křivosti paraboly *y* = *x*²

$$y = x^2$$
$$y' = 2x$$
$$y'' = 2$$

$$k = \frac{2}{\sqrt{(1+4x^2)^3}}$$

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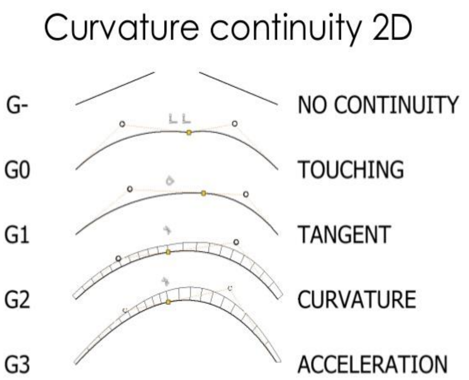
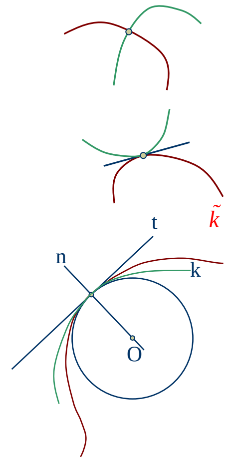
Order of continuity

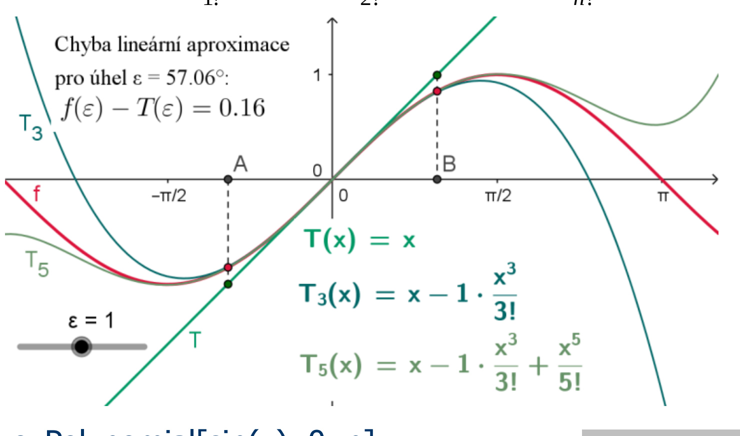
- Quantity of touching, smooth connection between two curves





Taylor approximation y=sin(x)

$$T(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

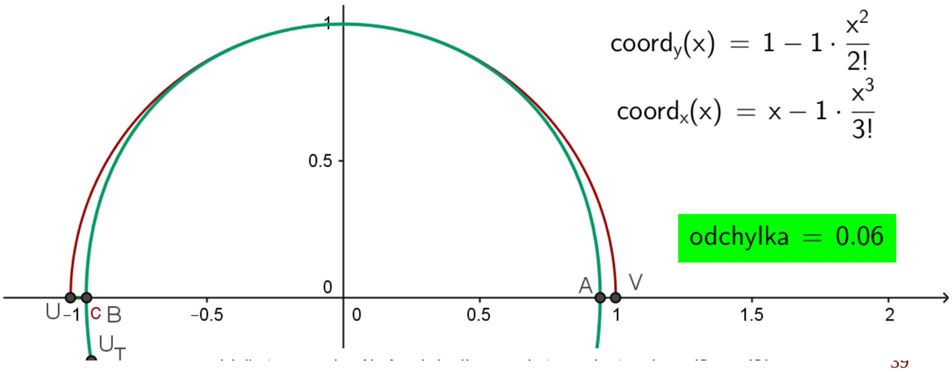


TaylorovPolynomial[sin(x), 0, n]

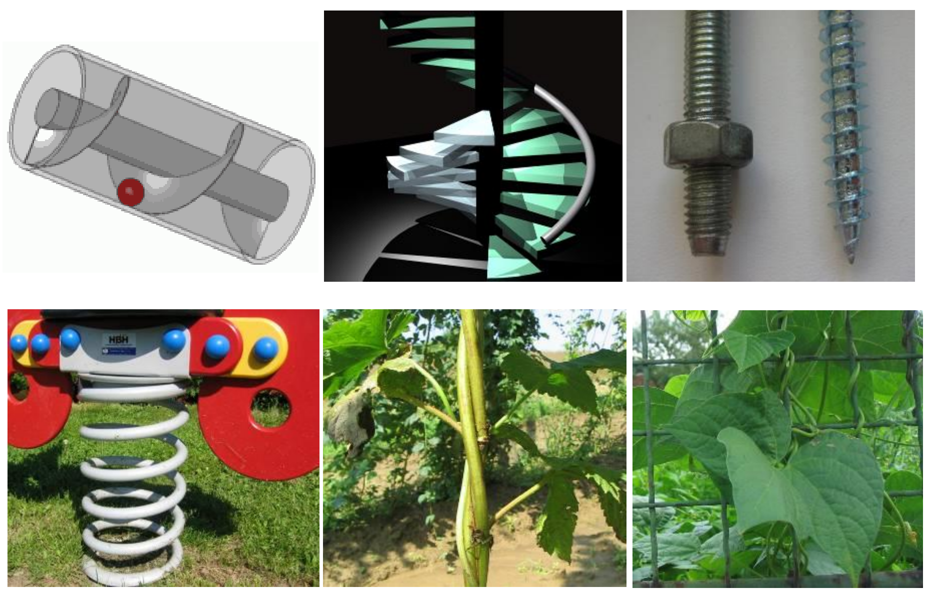
Taylor_sin.ggb

Cubic Taylor approximation of circle
c(t) = (sin *t*, cos *t*)

The components of the vector function are just the k-th degree Taylor polynomials of the components



Helix

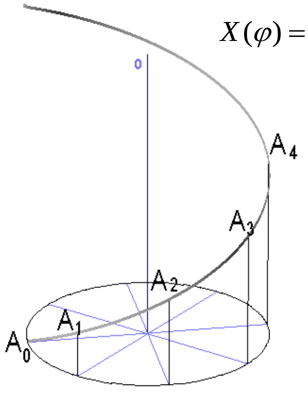


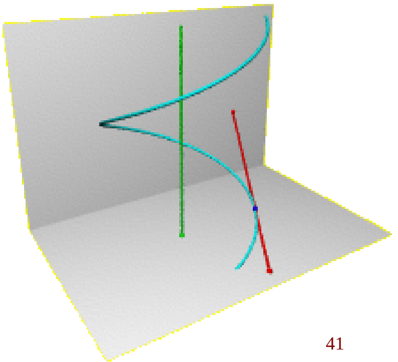
Circular Helix
– trajectory in the screw motion

Rotation about axis simultaneously with translation along the same axis.

Helix is determined by r, rate v_0 and axis $o = z$.

$$X(\varphi) = [r \cdot \cos\varphi; r \cdot \sin\varphi; v_0 \cdot \varphi]$$





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Tangent line to the helix

Helix: $X = [r \cdot \cos\varphi, r \cdot \sin\varphi, v_0 \varphi]$

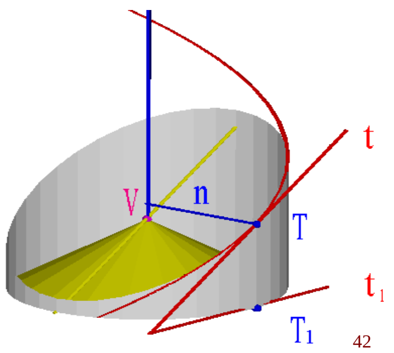
tangent vektor: $t = [-r \cdot \sin\varphi, r \cdot \cos\varphi, v_0]$

Top view of t: $t_1 = [-r \cdot \sin\varphi, r \cdot \cos\varphi, 0]$

Gradient (slope): $\tan\alpha = \frac{v_0}{\|t_1\|} = \frac{v_0}{r}$

Constant slope

It has the property that the tangent line at any point makes a constant angle with a fixed line called the axis.



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Arc length reparametrization

$$x = r \cdot \cos\varphi \qquad\qquad t = \left[-r \cdot \sin\varphi, r \cdot \cos\varphi, v_0\right]$$

$$y = r \cdot \sin\varphi \qquad\qquad \|t\| = \sqrt{r^2 + v_0^2}$$

$$z = v_0 \cdot \varphi \quad ;\varphi \in R$$

$$l = \int_0^{\varphi} \|\vec{t}\| \; d\varphi = \int_0^{\varphi} \sqrt{r^2 + v_0^2}\, d\varphi = \varphi\sqrt{r^2 + v_0^2} \Rightarrow \varphi = \frac{l}{\sqrt{r^2 + v_0^2}}$$

$$x = r \cdot \cos\frac{l}{\sqrt{r^2 + v_0^2}}$$

$$y = r \cdot \sin\frac{l}{\sqrt{r^2 + v_0^2}}$$

$$z = v_0 \cdot \frac{l}{\sqrt{r^2 + v_0^2}} \quad ;s \in R$$

Curvature of the helix

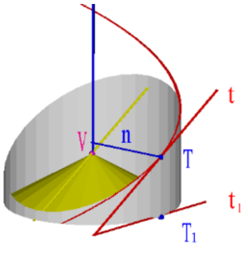
$$X(l) = \left[r\cos\left(\frac{l}{\sqrt{r^2+v_0^2}}\right); r\sin\left(\frac{l}{\sqrt{r^2+v_0^2}}\right); \frac{v_0 l}{\sqrt{r^2+v_0^2}}\right]$$

$$X'(l) = \left[\frac{-r}{\sqrt{r^2+v_0^2}} \cdot \sin\left(\frac{l}{\sqrt{r^2+v_0^2}}\right); \frac{r}{\sqrt{r^2+v_0^2}} \cdot \cos\left(\frac{l}{\sqrt{r^2+v_0^2}}\right); \frac{v_0}{\sqrt{r^2+v_0^2}}\right]$$

$$X''(l) = \left[\frac{-r}{r^2+v_0^2} \cdot \cos\left(\frac{l}{\sqrt{r^2+v_0^2}}\right); \frac{-r}{r^2+v_0^2} \cdot \sin\left(\frac{l}{\sqrt{r^2+v_0^2}}\right); 0\right]$$

$$k = \|X''(l)\| = \frac{r}{r^2+v_0^2}$$

Helix has a constant curvature



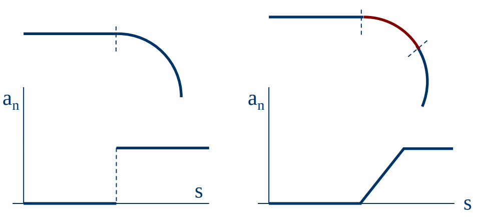
$$t = X'(l); \ \|t\| = 1$$

$$n = \frac{X''(l)}{k} = \left[-\cos\left(\frac{l}{\sqrt{r^2+v_0^2}}\right); -\cdot\sin\left(\frac{l}{\sqrt{r^2+v_0^2}}\right); 0\right]$$

Track transition curve

Track transition curve (spiral easement) is designed to prevent sudden changes in lateral (or centripetal) acceleration.

The start of the transition of the horizontal curve is at infinite radius and at the end of the transition it has the same radius as the curve itself, thus forming a very broad spiral.



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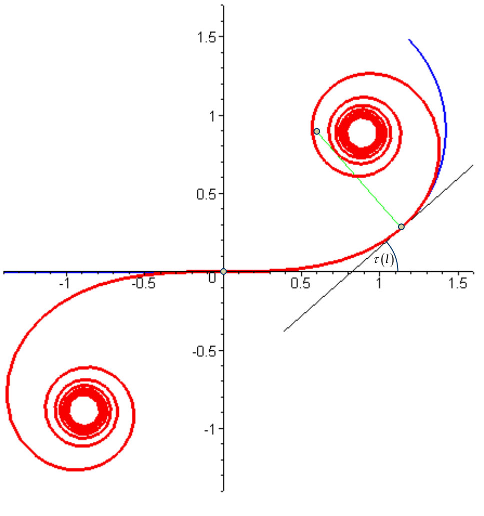
Clothoid – Euler spiral

- Curvature is proportional tu the arc length *k(l) = a.l*



$$X'(l) = (\cos\tau(l), \sin\tau(l))$$

$$X''(l) = (-\tau'(l)\cdot\sin\tau(l), \tau'(l)\cdot\cos\tau(l))$$

$$a\cdot l = k = \left\|\vec{X}''(l)\right\| = \left|\tau'(l)\right|$$

$$\tau(l) = \frac{a\cdot l^2}{2}$$

$$x(l) = \int_0^l \cos\frac{at^2}{2}\,dt$$

$$y(l) = \int_0^l \sin\frac{at^2}{2}\,dt$$

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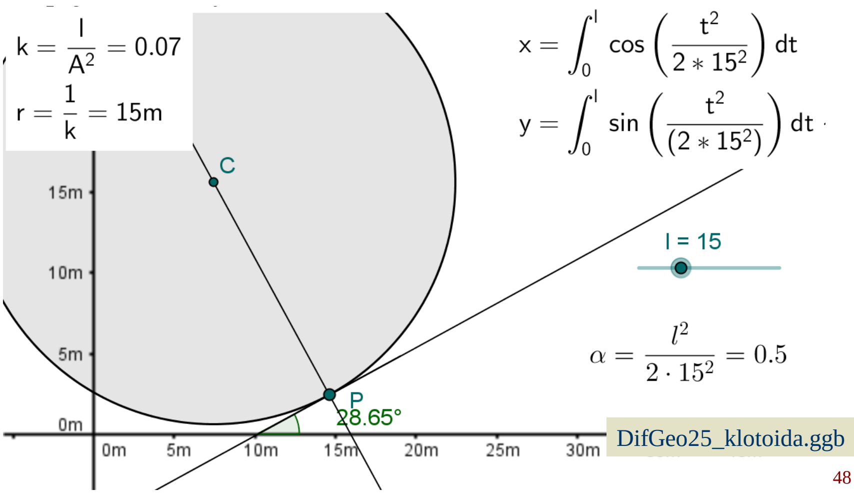
Determine the length of transition curve k = l / 225 used between the straight section and circular arc with radius r = 15 m



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Clothoid and cubic approximation

- Determine Taylor cubic approximation at point X(0) = [0,0].

$$T(t) = X(t_0) + \frac{X'(t_0)}{1!}(t-t_0) + \frac{X''(t_0)}{2!}(t-t_0)^2 + \ldots + \frac{X^{(n)}(t_0)}{n!}(t-t_0)^n$$

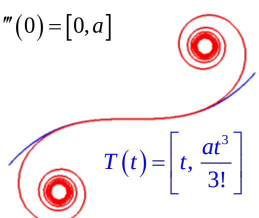
$$X(l) = \left[\int_0^l \cos\frac{at^2}{2}dt; \int_0^l \sin\frac{at^2}{2}dt\right] \qquad X(0) = [0,0]$$

$$X'(l) = \left[\cos\frac{al^2}{2}; \sin\frac{al^2}{2}\right] \qquad X'(0) = [1,0]$$

$$X''(l) = \left[-al \cdot \sin\frac{al^2}{2}; al\cos\frac{al^2}{2}\right] \qquad X''(0) = [0,0]$$

$$X'''(l) = \left[-a\sin\frac{al^2}{2} - a^2l^2\cos\frac{al^2}{2}; a\cos\frac{al^2}{2} - a^2l^2\sin\frac{al^2}{2}\right]; X'''(0) = [0,a]$$

$$T(t) = [0,0] + \frac{[1,0]}{1!}(t-0) + \frac{[0,0]}{2!}(t-0)^2 + \frac{[0,a]}{3!}(t-0)^3$$



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